

STAT/MA 41600
 In-Class Problem Set #2: August 29, 2014
 Solutions by Mark Daniel Ward

1a. Let A_j denote the event that the j th die has value 2. Then $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3) = 1/6 + 1/6 + 1/6 - (1/6)^2 - (1/6)^2 - (1/6)^2 + (1/6)^3 = 91/216$. [[To double-check: The probability a specific die does not show 2 is $5/6$. The probability none of the three dice show 2 is $(5/6)^3$. So the probability that at least one of the three dice show 2 is $1 - (5/6)^3 = 91/216$.]]

1b. The probability that no 2's appear is $(5/6)^3$. For exactly one value of 2, there are three disjoint ways to choose which die it will appear on, and then the probability is $(5/6)^2(1/6)$. So the desired probability is $(5/6)^3 + (3)(5/6)^2(1/6) = 25/27$.

2. Always true!! By inclusion-exclusion, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, but $P(A \cap B) \geq 0$, so $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$.

3a. If we place one red plate on the table, the second red plate has 3 equally likely places it could go, 2 of which are adjacent to the first red plate. So the red plates are adjacent with probability $2/3$ (and the blue plates will automatically be adjacent in such a case).

3b. Without loss of generality, place one red plate on the table. If all three red plates are to be in a cluster, this first red plate could be the far left, the middle, or the far right of the three in the eventual cluster (i.e., 3 possibilities). Each such possibility has probability $(2/5)(1/4) = 2/20$ of occurring. So the total probability is $2/20 + 2/20 + 2/20 = 6/20 = 3/10$. [[Alternative method: Place one red plate on the table. Then there are $\binom{5}{3} = 10$ equally likely ways remaining for the blue plates; in 3 of these ways, the blue plates are adjacent (and therefore the red plates are adjacent too), so the probability is $3/10$.]]

4a. There are $(10)(9)(8) = 10!/7!$ equally likely ways to pick, keeping track of order. Exactly $(10)(8)(6) = (2^3)(5)(4)(3) = (2^3)5!/2!$ ways yield unique colors. So the probability is $\frac{(2^3)5!7!}{2!10!} = 2/3$. [[Without keeping track of order, the probability is again $\binom{5}{3}2^3/\binom{10}{3} = 2/3$.]]

4b. There are $(16)(15)(14)(13)(12) = 16!/11!$ equally likely ways to pick, keeping track of order. Exactly $(16)(14)(12)(10)(8) = (2^5)(8)(7)(6)(5)(4) = (2^5)8!/3!$ ways yield unique colors. So the probability is $\frac{(2^5)8!11!}{3!16!} = \frac{16}{39}$. [[Without keeping track of order, $\binom{8}{5}2^5/\binom{16}{5} = \frac{16}{39}$.]]

4c. There are $(2s)(2s-1) \cdots (2s-r+1) = (2s)!/(2s-r)!$ equally likely ways to pick r socks, keeping track of order. Exactly $(2s)(2s-2) \cdots (2s-2(r-1)) = (2^r)(s)(s-1) \cdots (s-r+1) = (2^r)s!/(s-r)!$ ways yield unique colors. So the probability (of getting r uniquely colored socks) is $\frac{(2^r)s!(2s-r)!}{(s-r)!(2s)!}$. If you want $2r$ uniquely colored socks (i.e., r pairs), change all r to $2r$.

5a. There are 3 equally likely ways to paint all rocks the same color; the probability is $3/729$.

5b. There are 3 ways to pick two colors. With these two colors only, there are $2^6 = 64$ ways to paint all the rocks, but 2 of these 64 ways only actually use 1 of the 2 colors. So there are 62 ways to use *both* of the 2 selected colors. So the desired probability is $(3)(62)/729 = 186/729$.

5c. Since the probabilities must sum to 1, the probability that all 3 colors of paint are used is $1 - 3/729 - 186/729 = 540/729$.

6a. We have $\sum_{j=3}^{\infty} a^j = a^3 + a^4 + a^5 + a^6 + a^7 + \cdots = a^3(1 + a + a^2 + a^3 + a^4 \cdots) = \frac{a^3}{1-a}$.

6b. We have $\sum_{j=r}^{\infty} a^j = a^r + a^{r+1} + a^{r+2} + a^{r+3} + \cdots = a^r(1 + a + a^2 + a^3 + a^4 \cdots) = \frac{a^r}{1-a}$.