STAT/MA 41600
In-Class Problem Set #3: September 3, 2014
Solutions by Mark Daniel Ward

1. Events A and B are independent. Why? We note $P(A) = 4/8 = 1/2$ and $P(B) = 4/8 = 1/2$, and $P(A \cap B) = P(\{c, d\}) = 2/8 = 1/4$, so $P(A)P(B) = (1/2)(1/2) = 1/4 = P(A \cap B)$.
2. Events A and B are dependent. Why? We have $P(A) = 1/7$ and $P(B) = 1/7$, but $P(A \cap B) = (1/7)(1/5) \neq P(A)P(B)$.
3a. Using a red result as "good"—which has probability $r/(r + b + g + y)$, and a green or blue result as "bad"—which has probability $(b + g)/(r + b + g + y)$ (and yellow as neutral), it follows that the probability of the first "good" result coming sometime before the first "bad" result is $r/(r + b + g)$.
3b. This is the complement of **3a**, i.e., $1 - r/(r + b + g) = (g + b)/(r + b + g)$.
3c. Three or more yellows are chosen before a decision is made, if and only if the first three results are yellow, which happens with probability $(y/(r + b + g + y))^3$.
4. Events A and B are independent. Why? We note $P(A) = 18/36 = 1/2$ and $P(B) = 3/6 = 1/2$, and $P(A \cap B) = 9/36 = 1/4$, so $P(A)P(B) = (1/2)(1/2) = 1/4 = P(A \cap B)$.
5. Using sum of 2 as "good"—which has probability 1/24, and sum of 7 as "bad"—which has probability 4/24 (and the other possibilities as neutral), it follows that the probability of the first "good" result coming sometime before the first "bad" result is $1/(1 + 4) = 1/5$.
6a. The "brute force" method is to compute the probabilities by hand; write (a, b) for the outcome, with Alice's result in the first coordinate, and Bob's in the second coordinate. So

$$
\begin{aligned}
&P(\{(4,0),(4,1),(4,2),(4,3),(3,0),(3,1),(3,2),(2,0),(2,1),(1,0)\}) \\
&\qquad = \frac{1}{16}\cdot\frac{1}{8} + \frac{1}{16}\cdot\frac{3}{8} + \frac{1}{16}\cdot\frac{3}{8} + \frac{1}{16}\cdot\frac{1}{8} + \frac{4}{16}\cdot\frac{1}{8} + \frac{4}{16}\cdot\frac{3}{8} + \frac{4}{16}\cdot\frac{3}{8} + \frac{6}{16}\cdot\frac{1}{8} + \frac{6}{16}\cdot\frac{3}{8} + \frac{4}{16}\cdot\frac{1}{8} \\
&\qquad = 1/2
\end{aligned}
$$

Of course, it is not necessary to solve **6a** in such a painful way, if you are able to solve **6b** instead, which is much more general:
6b. Symmetry plays a key part here. This solution is immediate to see, if you just draw a picture. It's a bit harder to explain with words, but we will try:

After the first n flips, let A be the event Alice is strictly ahead, B be the event Bob is strictly ahead, or C be the event they are tied. So the probability that Alice ultimately has more heads is $1P(A) + 0P(B) + \frac{1}{2}P(C)$. [Notice $P(A) = P(B)$ since the coins are fair, so $P(A) = \frac{1}{2}P(A) + \frac{1}{2}P(A) = \frac{1}{2}P(A) + \frac{1}{2}P(B)$. Also notice $P(A) + P(B) + P(C) = 1$.] So the desired probability is $\frac{1}{2}P(A) + \frac{1}{2}P(B) + \frac{1}{2}P(C) = 1/2$.