STAT/MA 41600
In-Class Problem Set #4: September 5, 2014
Solutions by Mark Daniel Ward

1. Let A denote the event that the two results are equal; let B denote the event that the result on the red die is less than or equal to the result on the green die. Then $P(A \mid B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{6/36}{21/36} = 6/21 = 2/7$.
2. Given that A^c occurred, there remain 6 marbles: two (matched) pairs, and two individual (unmatched) marbles. So $P(B \mid A^c) = (4/6)(1/5) = 4/30 = 2/15$, because Bob has a 4/6 chance of getting one of the 4 marbles from the two pairs on his first draw, and then a 1/5 chance of getting its match on his second draw.
3a. Let A be the event that the ball on the 6th draw was *originally* white; let B be the event that the ball on the 6th draw was never previously drawn. Then we want $P(A \cap B)$, but A and B are independent, so we want $P(A)P(B)$. We have $P(A) = 3/10$ and $P(B) = (9/10)^5$. So $P(A \cap B) = (3/10)(9/10)^5$.
3b. Exactly the same idea, but we change "6th" to "16th," so $P(A \cap B) = (3/10)(9/10)^{15}$.
3c. Exactly the same idea, but we change "6th" to "76th," so $P(A \cap B) = (3/10)(9/10)^{75}$.
4a. Let A_j be the event that the jth person gets the proper coat. The probability *at least one* person gets their correct coat is $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3) = 1/3 + 1/3 + 1/3 - (1/3)(1/2) - (1/3)(1/2) - (1/3)(1/2) + (1/3)(1/2)(1)$, which we can simplify to $P(A_1 \cup A_2 \cup A_3) = 1 - 1/2 + 1/6$. So the desired probability is $1 - P(A_1 \cup A_2 \cup A_3) = 1 - 1 + 1/2 - 1/6 = 1/3$.
4b. Again, let A_j be the event that the jth person gets the proper coat. The probability *at least one* person gets their correct coat is $P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5)$, which has $\binom{5}{1} = 5$ terms $1/5$, and has $\binom{5}{2} = \frac{(5)(4)}{2}$ terms $-(1/5)(1/4)$, and has $\binom{5}{3} = \frac{(5)(4)(3)}{(3)(2)(1)}$ terms $(1/5)(1/4)(1/3)$, and has $\binom{5}{4} = \frac{(5)(4)(3)(2)}{(4)(3)(2)(1)}$ terms $-(1/5)(1/4)(1/3)(1/2)$, and 1 term $(1/5)(1/4)(1/3)(1/2)(1)$. So

$$P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5) = 1 - 1/2 + 1/6 - 1/24 + 1/120 = 1/1! - 1/2! + 1/3! - 1/4! + 1/5!,$$

and the desired probability is the complement, i.e.,

$$1 - P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5) = 1/0! - 1/1! + 1/2! - 1/3! + 1/4! - 1/5! = 11/30.$$

4c. By the same reasoning, the desired probability is

$$1 - P(A_1 \cup A_2 \cup \cdots \cup A_{10}) = 1/0! - 1/1! + 1/2! - 1/3! + \cdots - 1/9! + 1/10!.$$

4d. For a large number of people, the probability approaches $\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} = e^{-1} = 0.367879\ldots$.
5. The first time that either one of them get a value "1", there are 9 equally likely possibilities: $(1,1), (1,2), (1,3), (1,4), (2,1), (3,1), (4,1), (5,1), (6,1)$, where the first coordinate is Alice's result, and the second coordinate is Bob's result, so the desired probability is 1/9.
6. Let A be the event that all three dice show "2"; let B be the event that at least one dice shows "2". Then $P(A \mid B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{P(A)}{1-P(B^c)}$. We have $P(A) = (1/6)^3 = 1/216$ and $P(B^c) = (5/6)^3 = 125/216$. So $P(A \mid B) = \frac{1/216}{1-125/216} = 1/91$.