STAT/MA 41600
In-Class Problem Set #5: September 8, 2014
Solutions by Mark Daniel Ward

1. Let A be the event that the random student lives in a residence hall, and let B be the event that the student arrived on-time. Then $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A)+P(B|A^c)P(A^c)} = \frac{(.85)(.40)}{(.85)(.40)+(.70)(.60)} = 17/38 = 0.4474$.

2a. We compute $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A)+P(B|A^c)P(A^c)} = \frac{(\frac{1}{5})(\frac{1}{7})}{(\frac{1}{5})(\frac{1}{7})+(\frac{4}{6})(\frac{1}{5})(\frac{6}{7})} = 1/5$.
2b. We could either use the fact that $P(A^c|B) = 1 - P(A|B) = 1 - 1/5 = 4/5$, or we could compute directly: $P(A^c|B) = \frac{P(A^c\cap B)}{P(B)} = \frac{P(B|A^c)P(A^c)}{P(B|A)P(A)+P(B|A^c)P(A^c)} = \frac{(\frac{4}{6})(\frac{1}{5})(\frac{6}{7})}{(\frac{1}{5})(\frac{1}{7})+(\frac{4}{6})(\frac{1}{5})(\frac{6}{7})} = 4/5$.

3a. Let A be the event that the chosen ball was originally blue. Let B be the event that the chosen ball (at the start of round 2) was blue. Then $P(A|B) = \frac{P(A\cap B)}{P(B)}$. We note $P(A\cap B) = 7/10$ because $A\cap B$ happens if and only if one of the 7 originally blue balls is chosen at the start of round 2. To compute $P(B)$, we first define C as the event that the ball chosen in round 1 was blue. Then Also $P(B) = P(B|C)P(C) + P(B|C^c)P(C^c) = (7/10)(7/10) + (8/10)(3/10) = 73/100$. Thus $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{7/10}{73/100} = 70/73 = 0.9589$.
3b. We could either use the fact that $P(A^c|B) = 1 - P(A|B) = 1 - 70/73 = 3/73$, or we could compute directly: We have $P(A^c|B) = \frac{P(A^c\cap B)}{P(B)}$. We already computed $P(B) = 73/100$. Also, $A^c \cap B$ occurs if and only if the ball chosen at the start of round 2 is blue but was originally white. This happens with probability $(3/10)(1/10) = 3/100$. Thus $P(A^c|B) = \frac{P(A^c\cap B)}{P(B)} = \frac{3/100}{73/100} = 3/73 = 0.0411$.

4. Let A denote the probability that the second roll shows a value of "2". Let B denote the event that Bob is alive in the second part of the game, i.e., that Alice quits. Then B^c denotes the event that Alice is alive in the second part of the game, i.e., that Bob quits. We have $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) = (1/4)(6/24) + (1/6)(18/24) = 3/16 = 0.1875$.

5a. Let A_8, A_{12}, A_{20} denote (respectively) the events that the 8, 12, or 20 sided die was chosen. Let B denote the event that a 7 appears. Then we compute $P(A_8|B) = \frac{P(A_8\cap B)}{P(B)} = \frac{P(B|A_8)P(A_8)}{P(B|A_8)P(A_8)+P(B|A_{12})P(A_{12})+P(B|A_{20})P(A_{20})} = \frac{(1/8)(1/5)}{(1/8)(1/5)+(1/12)(1/5)+(1/20)(1/5)} = 15/31$.
5b and 5c. We compute $P(A_{12}|B) = \frac{P(A_{12}\cap B)}{P(B)} = \frac{P(B|A_{12})P(A_{12})}{P(B|A_8)P(A_8)+P(B|A_{12})P(A_{12})+P(B|A_{20})P(A_{20})} = \frac{(1/12)(1/5)}{(1/8)(1/5)+(1/12)(1/5)+(1/20)(1/5)} = 10/31$, and similarly, we compute $P(A_{20}|B) = \frac{P(A_{20}\cap B)}{P(B)} = \frac{P(B|A_{20})P(A_{20})}{P(B|A_8)P(A_8)+P(B|A_{12})P(A_{12})+P(B|A_{20})P(A_{20})} = \frac{(1/20)(1/5)}{(1/8)(1/5)+(1/12)(1/5)+(1/20)(1/5)} = 6/31$.

6. Let B be the event that the side facing upward is white. Let A be the event that the side facing downward is white. Then $P(A|B) = \frac{P(A\cap B)}{P(B)}$. We note $P(A\cap B) = 1/2$, since $A\cap B$ occurs if and only if the coin painted white on both sides is the coin that is chosen. We also note that $P(B) = 3/4$, since there are 4 equally likely sides that can be faces upward, and 3 of these sides are white. So $P(A|B) = \frac{1/2}{3/4} = 2/3$.

[[Moreover, this makes intuitive sense: There are 3 equally-likely white sides that can appear facing upward, and 2 of these possibilities correspond to the side facing downward being white too. So the probability is 2/3.]]