STAT/MA 41600

In-Class Problem Set #12: September 22, 2014

1. At a certain college, 40% of the students live in a residence hall (on-campus), and the other 60% of the students live off-campus. Suppose 6 students are independently selected at random, and X denotes the number of those students who live in a residence hall (on-campus).

1a. Find $\mathbb{E}(X^2)$ using the probability mass function of X.

1b. Find $\mathbb{E}(X^2)$ in a different way, namely, using the fact that $X = X_1 + \cdots + X_6$, where the X_j's are independent indicators. Expand $(X_1 + \cdots + X_6)^2$ into 36 terms, where 30 of them will behave one way, and the other 6 will behave another way.

1c. Find $\text{Var}(X)$ using your answer to $\mathbb{E}(X^2)$ and your answer to $\mathbb{E}(X)$ from last week.

1d. Find $\text{Var}(X)$ in a different manner, namely, by writing $X = X_1 + \cdots + X_6$ for some independent indicators, find $\text{Var}(X_j)$ for each j, then using the fact that the variance of the sum of independent random variables equals the sum of the variances.

2. When rolling a die, a "high value" is a 5 or 6. Roll seven dice. Let X denote the number of "high values" obtained on the seven dice altogether. Find $\mathbb{E}(X)$.

2a. Find $\mathbb{E}(X^2)$ using the probability mass function of X.

2b. Find $\mathbb{E}(X^2)$ in a different way, namely, using the fact that $X = X_1 + \cdots + X_7$, where the X_j's are independent indicators. Expand $(X_1 + \cdots + X_7)^2$ into 49 terms, where 42 of them will behave one way, and the other 7 will behave another way.

2c. Find $\text{Var}(X)$ using your answer to $\mathbb{E}(X^2)$ and your answer to $\mathbb{E}(X)$ from last week.

2d. Find $\text{Var}(X)$ in a different manner, namely, by writing $X = X_1 + \cdots + X_7$ for some independent indicators, find $\text{Var}(X_j)$ for each j, then using the fact that the variance of the sum of independent random variables equals the sum of the variances.

3A. Suppose Alice takes 3 cookies (without replacement) from a cookie jar that contains 5 cookies, 3 of which are chocolate, and the other 2 are non-chocolate. Let X be the number of chocolate cookies she gets. Find $\mathbb{E}(X)$.

3a. Find $\mathbb{E}(X^2)$ using the probability mass function of X.

3b. Find $\mathbb{E}(X^2)$ in a different way, namely, using the fact that $X = X_1 + X_2 + X_3$, where the X_j's are **dependent** indicators. Expand $(X_1 + X_2 + X_3)^2$ into 9 terms, where 6 of them will behave one way, and the other 3 will behave another way.

3c. Find $\text{Var}(X)$ using your answer to $\mathbb{E}(X^2)$ and your answer to $\mathbb{E}(X)$ from last week.

3B. Same type of question, but with replacement: Suppose Alice takes 3 cookies (with replacement) from a cookie jar that contains 5 cookies, 3 of which are chocolate, and the other 2 are non-chocolate. Let Y be the number of chocolate cookies she gets. Find $\mathbb{E}(Y)$.

3d. Find $\mathbb{E}(Y^2)$ using the probability mass function of X.

3e. Find $\mathbb{E}(Y^2)$ in a different way, namely, using the fact that $Y = Y_1 + Y_2 + Y_3$, where the Y_j's are independent indicators. Expand $(Y_1 + Y_2 + Y_3)^2$ into 9 terms, where 6 of them will behave one way, and the other 3 will behave another way.

3f. Find $\text{Var}(Y)$ using your answer to $\mathbb{E}(Y^2)$ and your answer to $\mathbb{E}(Y)$ from last week.

3g. Find $\text{Var}(Y)$ in a different manner, namely, by writing $Y = Y_1 + Y_2 + Y_3$ for some independent indicators, find $\text{Var}(Y_j)$ for each j, then using the fact that the variance of the sum of independent random variables equals the sum of the variances.

4. Suppose that a drawer contains 8 marbles: 2 are red, 2 are blue, 2 are green, and 2 are yellow. The marbles are rolling around in a drawer, so that all possibilities are equally likely when they are drawn. Suppose that a person removes marbles from the drawer, one at a time, without replacement, and she stops when red is selected for the first time. Let X denote the number of marbles removed, until red is selected for the first time. Find the expected value of X.

4a. Find $\mathbb{E}(X^2)$ using the probability mass function of X.

part 4b is an optional "bonus" for $+1/2$ point:

4b. Find $\mathbb{E}(X^2)$ in a different way, namely, using the fact that $X = X_1 + \ldots + X_7$, where the X_j's are **dependent** indicators. Expand $(X_1 + \ldots + X_7)^2$ into 49 terms. [[Hint: If you define $X_j = 1$ when $X \geq j$ and $X_j = 0$ otherwise, then notice that $X_i X_j = X_j$ for any $i \leq j$, so you can simplify the 49 terms down to just a small number of terms.]]

4c. Find $\mathrm{Var}(X)$ using your answer to $\mathbb{E}(X^2)$ and your answer to $\mathbb{E}(X)$ from last week.

5. Suppose Alice rolls a 6-sided die, and Bob rolls a 4-sided die. Let X denote the *minimum* value on the two dice. Find $\mathbb{E}(X)$.

5a. Find $\mathbb{E}(X^2)$ using the probability mass function of X.

part 5b is an optional "bonus" for $+1/2$ point:

5b. Find $\mathbb{E}(X^2)$ in a different way, namely, using the fact that $X = X_1 + X_2 + X_3 + X_4$, where the X_j's are **dependent** indicators. Expand $(X_1 + X_2 + X_3 + X_4)^2$ into 16 terms. [[Hint: If you define $X_j = 1$ when $X \geq j$ and $X_j = 0$ otherwise, then notice that $X_i X_j = X_j$ for any $i \leq j$, so you can simplify the 16 terms down to just a small number of terms.]]

5c. Find $\mathrm{Var}(X)$ using your answer to $\mathbb{E}(X^2)$ and your answer to $\mathbb{E}(X)$ from last week.

6. Six rocks are sitting in a straight line. We paint them, using up to three colors (say, R's, W's, and B's). Suppose all of the $3^6 = 729$ outcomes are equally likely. Let X denote the number of colors used altogether to paint the rocks. Find $\mathbb{E}(X)$.

6a. Find $\mathbb{E}(X^2)$ using the probability mass function of X.

part 6b is an optional "bonus" for $+1/2$ point:

6b. Find $\mathbb{E}(X^2)$ in a different way, namely, using the fact that $X = X_1 + X_2 + X_3$, where the X_j's are **dependent** indicators. Expand $(X_1 + X_2 + X_3)^2$ into 9 terms, where 6 of them will behave one way, and the other 3 will behave another way.

6c. Find $\mathrm{Var}(X)$ using your answer to $\mathbb{E}(X^2)$ and your answer to $\mathbb{E}(X)$ from last week.