

STAT/MA 41600  
In-Class Problem Set #12: September 22, 2014

**1.** At a certain college, 40% of the students live in a residence hall (on-campus), and the other 60% of the students live off-campus. Suppose 6 students are independently selected at random, and  $X$  denotes the number of those students who live in a residence hall (on-campus).

**1a.** Find  $\mathbb{E}(X^2)$  using the probability mass function of  $X$ .

**1b.** Find  $\mathbb{E}(X^2)$  in a different way, namely, using the fact that  $X = X_1 + \cdots + X_6$ , where the  $X_j$ 's are independent indicators. Expand  $(X_1 + \cdots + X_6)^2$  into 36 terms, where 30 of them will behave one way, and the other 6 will behave another way.

**1c.** Find  $\text{Var}(X)$  using your answer to  $\mathbb{E}(X^2)$  and your answer to  $\mathbb{E}(X)$  from last week.

**1d.** Find  $\text{Var}(X)$  in a different manner, namely, by writing  $X = X_1 + \cdots + X_6$  for some independent indicators, find  $\text{Var}(X_j)$  for each  $j$ , then using the fact that the variance of the sum of independent random variables equals the sum of the variances.

**2.** When rolling a die, a “high value” is a 5 or 6. Roll seven dice. Let  $X$  denote the number of “high values” obtained on the seven dice altogether. Find  $\mathbb{E}(X)$ .

**2a.** Find  $\mathbb{E}(X^2)$  using the probability mass function of  $X$ .

**2b.** Find  $\mathbb{E}(X^2)$  in a different way, namely, using the fact that  $X = X_1 + \cdots + X_7$ , where the  $X_j$ 's are independent indicators. Expand  $(X_1 + \cdots + X_7)^2$  into 49 terms, where 42 of them will behave one way, and the other 7 will behave another way.

**2c.** Find  $\text{Var}(X)$  using your answer to  $\mathbb{E}(X^2)$  and your answer to  $\mathbb{E}(X)$  from last week.

**2d.** Find  $\text{Var}(X)$  in a different manner, namely, by writing  $X = X_1 + \cdots + X_7$  for some independent indicators, find  $\text{Var}(X_j)$  for each  $j$ , then using the fact that the variance of the sum of independent random variables equals the sum of the variances.

**3A.** Suppose Alice takes 3 cookies (without replacement) from a cookie jar that contains 5 cookies, 3 of which are chocolate, and the other 2 are non-chocolate. Let  $X$  be the number of chocolate cookies she gets. Find  $\mathbb{E}(X)$ .

**3a.** Find  $\mathbb{E}(X^2)$  using the probability mass function of  $X$ .

**3b.** Find  $\mathbb{E}(X^2)$  in a different way, namely, using the fact that  $X = X_1 + X_2 + X_3$ , where the  $X_j$ 's are **dependent** indicators. Expand  $(X_1 + X_2 + X_3)^2$  into 9 terms, where 6 of them will behave one way, and the other 3 will behave another way.

**3c.** Find  $\text{Var}(X)$  using your answer to  $\mathbb{E}(X^2)$  and your answer to  $\mathbb{E}(X)$  from last week.

**3B.** Same type of question, but with replacement: Suppose Alice takes 3 cookies (with replacement) from a cookie jar that contains 5 cookies, 3 of which are chocolate, and the other 2 are non-chocolate. Let  $Y$  be the number of chocolate cookies she gets. Find  $\mathbb{E}(Y)$ .

**3d.** Find  $\mathbb{E}(Y^2)$  using the probability mass function of  $Y$ .

**3e.** Find  $\mathbb{E}(Y^2)$  in a different way, namely, using the fact that  $Y = Y_1 + Y_2 + Y_3$ , where the  $Y_j$ 's are independent indicators. Expand  $(Y_1 + Y_2 + Y_3)^2$  into 9 terms, where 6 of them will behave one way, and the other 3 will behave another way.

**3f.** Find  $\text{Var}(Y)$  using your answer to  $\mathbb{E}(Y^2)$  and your answer to  $\mathbb{E}(Y)$  from last week.

**3g.** Find  $\text{Var}(Y)$  in a different manner, namely, by writing  $Y = Y_1 + Y_2 + Y_3$  for some independent indicators, find  $\text{Var}(Y_j)$  for each  $j$ , then using the fact that the variance of the sum of independent random variables equals the sum of the variances.

4. Suppose that a drawer contains 8 marbles: 2 are red, 2 are blue, 2 are green, and 2 are yellow. The marbles are rolling around in a drawer, so that all possibilities are equally likely when they are drawn. Suppose that a person removes marbles from the drawer, one at a time, without replacement, and she stops when red is selected for the first time. Let  $X$  denote the number of marbles removed, until red is selected for the first time. Find the expected value of  $X$ .

4a. Find  $\mathbb{E}(X^2)$  using the probability mass function of  $X$ .

**part 4b is an optional “bonus” for +1/2 point:**

4b. Find  $\mathbb{E}(X^2)$  in a different way, namely, using the fact that  $X = X_1 + \dots + X_7$ , where the  $X_j$ 's are **dependent** indicators. Expand  $(X_1 + \dots + X_7)^2$  into 49 terms. [[Hint: If you define  $X_j = 1$  when  $X \geq j$  and  $X_j = 0$  otherwise, then notice that  $X_i X_j = X_j$  for any  $i \leq j$ , so you can simplify the 49 terms down to just a small number of terms.]]

4c. Find  $\text{Var}(X)$  using your answer to  $\mathbb{E}(X^2)$  and your answer to  $\mathbb{E}(X)$  from last week.

5. Suppose Alice rolls a 6-sided die, and Bob rolls a 4-sided die. Let  $X$  denote the *minimum* value on the two dice. Find  $\mathbb{E}(X)$ .

5a. Find  $\mathbb{E}(X^2)$  using the probability mass function of  $X$ .

**part 5b is an optional “bonus” for +1/2 point:**

5b. Find  $\mathbb{E}(X^2)$  in a different way, namely, using the fact that  $X = X_1 + X_2 + X_3 + X_4$ , where the  $X_j$ 's are **dependent** indicators. Expand  $(X_1 + X_2 + X_3 + X_4)^2$  into 16 terms. [[Hint: If you define  $X_j = 1$  when  $X \geq j$  and  $X_j = 0$  otherwise, then notice that  $X_i X_j = X_j$  for any  $i \leq j$ , so you can simplify the 16 terms down to just a small number of terms.]]

5c. Find  $\text{Var}(X)$  using your answer to  $\mathbb{E}(X^2)$  and your answer to  $\mathbb{E}(X)$  from last week.

6. Six rocks are sitting in a straight line. We paint them, using up to three colors (say,  $R$ 's,  $W$ 's, and  $B$ 's). Suppose all of the  $3^6 = 729$  outcomes are equally likely. Let  $X$  denote the number of colors used altogether to paint the rocks. Find  $\mathbb{E}(X)$ .

6a. Find  $\mathbb{E}(X^2)$  using the probability mass function of  $X$ .

**part 6b is an optional “bonus” for +1/2 point:**

6b. Find  $\mathbb{E}(X^2)$  in a different way, namely, using the fact that  $X = X_1 + X_2 + X_3$ , where the  $X_j$ 's are **dependent** indicators. Expand  $(X_1 + X_2 + X_3)^2$  into 9 terms, where 6 of them will behave one way, and the other 3 will behave another way.

6c. Find  $\text{Var}(X)$  using your answer to  $\mathbb{E}(X^2)$  and your answer to  $\mathbb{E}(X)$  from last week.