STAT/MA 41600
In-Class Problem Set #14/#15: September 24, 2014
(there is no Problem Set #13)
Solutions by Mark Daniel Ward

1a. We have $X_j = 1$ if and only if the first $j-1$ students live off-campus, which happens with probability $.60^{j-1}$; otherwise, $X_j = 0$. So $\mathbb{E}(X_j) = .60^{j-1}$.
1b. We calculate $\mathbb{E}(X) = \mathbb{E}(\sum_{j=1}^{\infty} X_j) = \sum_{j=1}^{\infty} \mathbb{E}(X_j) = \sum_{j=1}^{\infty} .60^{j-1} = \frac{1}{1-.60} = \frac{1}{.40} = 5/2$.

2a. We have $X_j = 1$ if and only if the first $j-1$ are not "high values", which happens with probability $(2/3)^{j-1}$; otherwise, $X_j = 0$. So $\mathbb{E}(X_j) = (2/3)^{j-1}$.
2b. We calculate $\mathbb{E}(X) = \mathbb{E}(\sum_{j=1}^{\infty} X_j) = \sum_{j=1}^{\infty} \mathbb{E}(X_j) = \sum_{j=1}^{\infty}(2/3)^{j-1} = \frac{1}{1-2/3} = \frac{1}{1/3} = 3$.

3a. It is true that $X = X_1 + X_2 + X_3 + X_4 + X_5$, where $X_j = 1$ if the jth adjacent pair of chairs show the same result, or $X_j = 0$ otherwise. BUT the X_j's are not a collection of five *independent* indicator random variables. For instance, if $X_1 = 1$, $X_2 = 1$, $X_3 = 1$, $X_4 = 1$, then we must have $X_5 = 1$ too (because all 5 flips must match in this case). An alternative way to realize this is to see that, if $X_1 = 0$, $X_2 = 0$, $X_3 = 0$, $X_4 = 0$, then we must have $X_5 = 1$ (because if the flips are strictly alternating as we move around the table, the first and last flips must match, since there are an odd number of flips). So the X_j's are not (as a collection) all independent. So X is not Binomial. [[In fact, it is an interesting exercise to find the probability mass function of X. You did not have to calculate it, but I will give it here, in case you are interested: The mass of X is $p_X(0) = 0$, $p_X(1) = 10/32$, $p_X(2) = 0$, $p_X(3) = 20/32$, $p_X(4) = 0$, $p_X(5) = 2/32$, which is not Binomial at all!!]]
3b. As above, let $X = X_1 + X_2 + X_3 + X_4 + X_5$, where $X_j = 1$ if the jth adjacent pair of chairs show the same result, or $X_j = 0$ otherwise. Then $\mathbb{E}(X_j) = 1/2$, so $\mathbb{E}(X) = \mathbb{E}(X_1 + X_2 + X_3 + X_4 + X_5) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) + \mathbb{E}(X_4) + \mathbb{E}(X_5) = 5(1/2) = 5/2$.
3c. Let $X = X_1 + X_2 + \cdots + X_{75}$, where $X_j = 1$ if the jth adjacent pair of chairs show the same result, or $X_j = 0$ otherwise. Then $\mathbb{E}(X_j) = 1/2$, so $\mathbb{E}(X) = \mathbb{E}(X_1 + X_2 + \cdots + X_5) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \cdots + \mathbb{E}(X_5) = 75(1/2) = 75/2$.

4. Let $X = X_1 + X_2 + X_3 + X_4 + X_5 + X_6$, where $X_j = 1$ if the jth adjacent pair of rocks show the same result, or $X_j = 0$ otherwise. Then $\mathbb{E}(X_j) = 1/3$, so $\mathbb{E}(X) = \mathbb{E}(X_1 + X_2 + X_3 + X_4 + X_5 + X_6) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) + \mathbb{E}(X_4) + \mathbb{E}(X_5) + \mathbb{E}(X_6) = 6(1/3) = 2$.

5. The mass of Z is $P(X = 0) = 6/36$; $P(X = 1) = 10/36$; $P(X = 2) = 8/36$; $P(X = 3) = 6/36$; $P(X = 4) = 4/36$; $P(X = 5) = 2/36$; So $\mathbb{E}(Z) = 0(6/36) + 1(10/36) + 2(8/36) + 3(6/36) + 4(4/36) + 5(2/36) = 70/36 = 35/18$. Alternatively, the expected value is $\mathbb{E}(|X - Y|) = \sum_{x=1}^{6} \sum_{y=1}^{6} \frac{1}{36}|x - y| = 35/18$.

6. We are given that there are exactly 5 rolls which are 9's or 11's. Given that a roll is 9 or 11, the probability it is a 9 is $\frac{4/36}{4/36+2/36} = 4/6 = 2/3$. Thus, given $X + Y = 5$, and knowing that the rolls are independent, the conditional distribution of X is Binomial with $n = 5$ and $p = 2/3$. In other words, $p_{X|X+Y}(x \mid 5) = \binom{5}{x}(2/3)^x(1/3)^{5-x}$ for $0 \le x \le 5$.