

STAT/MA 41600
 In-Class Problem Set #16: September 26, 2014
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1abcd. Since X is a Geometric random variable with $p = 1/6$ then $\mathbb{E}(X) = 1/p = 6$ and $\text{Var}(X) = q/p^2 = (5/6)/(1/6)^2 = 30$. Since $Y = X - 1$ then $\mathbb{E}(Y) = \mathbb{E}(X - 1) = \mathbb{E}(X) - 1 = 6 - 1 = 5$ and $\text{Var}(Y) = \text{Var}(X - 1) = \text{Var}(X) = 30$.

2. Matilda wins $1.25Y - 10$, so her expected net winnings are $\mathbb{E}(1.25Y - 10) = 1.25\mathbb{E}(Y) - 10 = 1.25(5) - 10 = -3.75$. In other words, her expected net loss is 3.75.

3. We have $P(a \leq X \leq b) = \sum_{j=a}^b q^{j-1}p = (q^{a-1} + q^a + \dots + q^{b-2} + q^{b-1})(1 - q) = q^{a-1} - q^b$.

4a. We have $P(X > b \mid X > a) = \frac{P(X > b \& X > a)}{P(X > a)} = \frac{P(X > b)}{P(X > a)} = \frac{q^b}{q^a} = q^{b-a}$.

4b. We have $P(X \leq b \mid X > a) = 1 - (X > b \mid X > a) = 1 - q^{b-a}$.

4c. We have $P(X = b \mid X > a) = \frac{P(X = b \& X > a)}{P(X > a)} = \frac{P(X = b)}{P(X > a)} = \frac{q^{b-1}p}{q^a} = pq^{b-1-a}$.

5ab. Since X is a Geometric random variable with $p = 1/24$ then $\mathbb{E}(X) = 1/p = 24$ and $\text{Var}(X) = q/p^2 = (23/24)/(1/24)^2 = 552$.

6. We have

$$\begin{aligned}
 P(X \geq y) &= \sum_{y=1}^{\infty} \sum_{x=y}^{\infty} p_{X,Y}(x, y) \\
 &= \sum_{y=1}^{\infty} \sum_{x=y}^{\infty} (5/6)^{x-1} (1/6) (1/2)^{y-1} (1/2) \\
 &= (1/6)(1/2) \sum_{y=1}^{\infty} (1/2)^{y-1} \sum_{x=y}^{\infty} (5/6)^{x-1} \\
 &= (1/6)(1/2) \sum_{y=1}^{\infty} (1/2)^{y-1} \frac{(5/6)^{y-1}}{1 - 5/6} \\
 &= (1/2) \sum_{y=1}^{\infty} ((1/2)(5/6))^{y-1} \\
 &= (1/2) \sum_{y=1}^{\infty} (5/12)^{y-1} \\
 &= (1/2)/(1 - 5/12) \\
 &= 6/7
 \end{aligned}$$

Here's an equivalent way to see this. We want the probability that the first head occurs earlier, or at the same time, as the first "1." Of the 12 equally-likely results (x, y) with $1 \leq x \leq 6$ and $y = H$ or $y = T$, there are 7 equally-likely results in which $x = 1$ or $y = H$ or both, namely: $(1, T)$, $(1, H)$, $(2, H)$, $(3, H)$, $(4, H)$, $(5, H)$, $(6, H)$. Given that one of these 7 equally-likely results occurs, it is one of the 6 results with H (not T) with probability 6/7.