STAT/MA 41600
In-Class Problem Set #18: October 1, 2014
Solutions by Mark Daniel Ward

1. Let X denote the number of raindrops that fall during the next one minute. Then X is Poisson with $\lambda = 6$.
1a. We have $P(X = 5) = e^{-6}6^5/5! = 0.1606$.
1b. We have $P(X = 0) = e^{-6}6^0/0! = 0.002479$.
1c. We have $P(X \geq 4) = 1 - P(X \leq 3) = 1 - e^{-6}6^0/0! - e^{-6}6^1/1! - e^{-6}6^2/2! - e^{-6}6^3/3! = 1 - 61e^{-6} = 0.8488$.

2. Let Y denote the number of raindrops that fall during the next 15 seconds. Then Y is Poisson with $\lambda = 1.5$.
2a. We have $P(Y = 1) = e^{-1.5}(1.5)^1/1! = 0.3347$.
2b. We have $P(Y = 0) = e^{-1.5}(1.5)^0/0! = 0.2231$.
2c. We have $P(Y \geq 3) = 1 - P(Y \leq 2) = 1 - e^{-1.5}(1.5)^0/0! - e^{-1.5}(1.5)^1/1! - e^{-1.5}(1.5)^2/2! = 1 - (29/8)e^{-3/2} = 0.1912$.

3. We have $P(X \geq 4) = 1 - P(X \leq 3) = 1 - e^{-\lambda}\lambda^0/0! - e^{-\lambda}\lambda^1/1! - e^{-\lambda}\lambda^2/2! - e^{-\lambda}\lambda^3/3! = 1 - \frac{1}{6}e^{-\lambda}(6 + 6\lambda + 3\lambda^2 + \lambda^3)$.

4a. The random variable X is a Binomial with $n = 10000$ and $p = 1/1000$. So the mass is $P(X = x) = p_X(x) = \binom{10000}{x}(1/1000)^x(999/1000)^{10000-x}$ for $0 \leq x \leq 10000$.
4b. We have $\mathbb{E}(X) = np = (10000)(1/1000) = 10$.
4c. We have $\text{Var}\,(X) = npq = (10000)(1/1000)(999/1000) = 999/100$.
4d. Since the distribution of X is approximately Poisson with $\lambda = np = 10$, we have $P(X = 9) \approx e^{-10}10^9/9! = 0.125$.

5a. Since the distribution of X is approximately Poisson with $\lambda = (10000)(3/10000) = 3$, then $P(X \geq 5) = 1 - P(X \leq 4) \approx 1 - e^{-3}3^0/0! - e^{-3}3^1/1! - e^{-3}3^2/2! - e^{-3}3^3/3! - e^{-3}3^4/4! = 1 - (131/8)e^{-3} = 0.1847$.
5b. Since the distribution of Y is approximately Poisson with $\lambda = (10000)(5/10000) = 5$, and since X and Y are independent, the distribution of $X+Y$ is roughly Poisson with $\lambda = 8$. Thus $P(8 \leq X + Y \leq 9) = P(X + Y = 8) + P(X + Y = 9) = e^{-8}8^8/8! + e^{-8}8^9/9! = 0.2637$.

6. We have $P(Y > X) = \sum_{x=0}^{\infty}\sum_{y=x+1}^{\infty}(e^{-\lambda}\lambda^x/x!)(q^{y-1}p) = \sum_{x=0}^{\infty}(e^{-\lambda}\lambda^x/x!)p\sum_{y=x+1}^{\infty}q^{y-1} = e^{-\lambda}\sum_{x=0}^{\infty}(\lambda^x/x!)pq^x/(1-q) = e^{-\lambda}\sum_{x=0}^{\infty}(\lambda q)^x/x! = e^{-\lambda}e^{\lambda q} = e^{\lambda(q-1)} = e^{-\lambda p}$.