STAT/MA 41600
In-Class Problem Set #19: October 3, 2014
Solutions by Mark Daniel Ward

1. Since X is a hypergeometric with $N = 15$, $M = 5$, and $n = 3$, then:
1a. The mass of X is $p_X(x) = P(X = x) = \binom{5}{x}\binom{10}{3-x}/\binom{15}{3}$.
1b. The expected value of X is $\mathbb{E}(X) = \frac{(3)(5)}{15} = 1$.
1c. The variance of X is $\mathrm{Var}\,(X) = (3)(5/15)(1 - 5/15)((15 - 3)/(15 - 1)) = 4/7$.

2. With the given conditions, if X is the number of "high values" among the 4 chosen dice, then X is a hypergeometric random variable with $N = 10$, $M = 8$, and $n = 4$, so $P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2)$, but $P(X = 0) = 0$ and $P(X = 1) = 0$, so $P(X \le 2) = P(X = 2) = \binom{8}{2}\binom{2}{2}/\binom{10}{4} = 2/15$.

3a. When choose 3 rocks without replacement, since the colors of the rocks must be independent (since all outcomes are equally likely), then the number of blue rocks is binomial with $n = 3$ and $p = 1/3$, so the desired probability is $\binom{3}{2}(1/3)^2(2/3)^1 + \binom{3}{3}(1/3)^3(2/3)^0 = 7/27$.

3b. The probability that 3 distinct rocks are selected is $(6/6)(5/6)(4/6) = 5/9$, and in that case, the probability that two or more of the 3 chosen rocks are blue is $7/27$, as in part **(3a)**.

The probability that 2 distinct rocks are selected is $(3)(6/6)(5/6)(1/6) = 5/12$, and in that case, the probability that two or more of the 3 chosen rocks are blue is $1/3$ (since this happens if and only if the rock that is selected twice is blue).

The probability that 1 rock is selected all three times is $(6/6)(1/6)(1/6) = 1/36$, and in that case, the probability that two or more of the 3 chosen rocks are blue is $1/3$.

So the desired probability is $(5/9)(7/27) + (5/12)(1/3) + (1/36)(1/3) = 71/243$.

4. We have

$$\begin{aligned}
P(X \ge Y) &= p_X(3) + (p_X(2) - p_{X,Y}(2,3)) + p_{X,Y}(1,1) + p_{X,Y}(1,0) + P_{X,Y}(0,0)\\
&= 1/27 + (2/9 - (2/9)(1/30)) + (4/9)(1/2) + (4/9)(1/6) + (8/27)(1/6)\\
&= 242/405
\end{aligned}$$

5. Since X is Hypergeometric with $N = 10, M = 8, n = 3$ and Y is Binomial with $n = 3$ and $p = 8/10$, then

$$\begin{aligned}
P(X = Y) &= P(X = Y = 0) + P(X = Y = 1) + P(X = Y = 2) + P(X = Y = 3)\\
&= (0)(1/125) + (1/15)(12/125) + (7/15)(48/125) + (7/15)(64/125)\\
&= 796/1875
\end{aligned}$$

6a. We have $\mathbb{E}(X^2) = 4\mathbb{E}(X_1^2) + 12\mathbb{E}(X_1X_2) = (4)(9/12) + (12)(9/12)(8/11) = 105/11$.
6b. Since X is Hypergeometric with $N = 12, M = 9, n = 4$, then $p_X(x) = \binom{9}{x}\binom{3}{4-x}/\binom{12}{4}$.
6c. We compute $\mathbb{E}(X^2) = \sum_{x=0}^{4} x^2\binom{9}{x}\binom{3}{4-x}/\binom{12}{4} = 105/11$.
6d. Since X is Hypergeometric with $N = 12, M = 9, n = 4$, then $\mathbb{E}(X) = nM/N = 3$.
6e. Since X is Hypergeometric with $N = 12, M = 9, n = 4$, then $\mathrm{Var}\,(X) = n(M/N)(1 - M/N)(N-n)/(N-1) = 6/11$. Or simply, $\mathrm{Var}\,(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 105/11 - 3^2 = 6/11$.