STAT/MA 41600
In-Class Problem Set #20/#22: October 6, 2014
(there is no Problem Set #21)
Solutions by Mark Daniel Ward

1a. Each X_j has $\mathbb{E}(X_j) = 1/5$, so $\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_{48}) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_{48}) = 48(1/5) = 48/5$.
1b. The expected value of X^2 is $\mathbb{E}(X^2) = \mathbb{E}((X_1 + \cdots + X_{48})^2)$, which can be expanded to have 48 terms that are each equal to $\mathbb{E}(X_1^2) = \mathbb{E}(X_1) = 1/5$, and the other $48^2 - 48$ terms are each equal to $\mathbb{E}(X_1X_2) = (1/5)(2/6) = 2/30$ (or, alternatively, $1/\binom{6}{2} = 2/30$). Thus $\mathbb{E}(X^2) = 48(1/5) + (48^2 - 48)(2/30) = 160$. So the variance of X is $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 160 - (48/5)^2 = 1696/25$.

2a. The probability we get 2 books from each subject is $\binom{5}{2}\binom{7}{2}\binom{10}{2}/\binom{22}{6} = 450/3553$.
2b. The probability we get 6 math books is $\binom{5}{6}/\binom{22}{6} = 0$; the probability we get 6 biology books is $\binom{7}{6}/\binom{22}{6} = 1/10659$; the probability we get 6 history books is $\binom{10}{6}/\binom{22}{6} = 10/3553$; so the desired probability is $0 + 1/10659 + 10/3553 = 31/10659$.

3. Let X_j indicate whether the jth group consists of 1 Statistics and 1 Mathematics student, i.e., $X_j = 1$ if there is 1 Statistics and 1 Mathematics student in the jth group, and $X_j = 0$ otherwise. Then $X_j = 1$ with probability 7/13, so $\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_7) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_7) = (7)(7/13) = 49/13$.

4a. Since X is Binomial with $n = 3$ and $p = 7/16$ (and $q = 9/16$), then $\mathrm{Var}(X) = npq = (3)(7/16)(9/16) = 189/256$.
4a. Since Y is Hypergeometric with $n = 3$ and $M = 7$ and $N = 16$, then $\mathrm{Var}(Y) = n(M/N)(1 - M/N)(N - n)/(N - 1) = (3)(7/16)(9/16)(13/15) = 819/1280$.

5a. The probability that their choices are unique is $(10/10)(9/10)(8/10)(7/10) = 63/125$.

5b. The probability that their choices are unique is (as before) $(10/10)(9/10)(8/10)(7/10) = 63/125$. Given that their choices are unique, the choices are in the correct order with probability $1/4! = 1/24$. So the desired probability is $(63/125)(1/24) = 21/1000$.

[[Alternatively: There are $10^4 = 10000$ equally likely ways that the numbers could be chosen; there are $\binom{10}{4} = 210$ ways that the numbers chosen can be distinct, and each such way has just 1 corresponding way in which the numbers are in the correct order. So the probability is $210/10000 = 21/1000$.]]

6a. We have $\mathbb{E}(X) = \mathbb{E}(2Y + 3) = 2\mathbb{E}(Y) + 3 = 2(9/2) + 3 = 12$.
6b. We have $\mathrm{Var}(X) = \mathrm{Var}(2Y + 3) = 2^2\,\mathrm{Var}(Y) = 2^2(8^2 - 1)/12 = 21$.