

STAT/MA 41600
In-Class Problem Set #26: October 20, 2014
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1a. No, X and Y are not independent, because they are not defined in rectangular regions.

1b. For $0 \leq x \leq 3$, we have $f_X(x) = \int_0^2 1/8 dy = 1/4$.

For $3 \leq x \leq 5$, we have $f_X(x) = \int_{x-3}^2 1/8 dy = (5-x)/8$.

1c. For $0 \leq y \leq 2$, we have $f_Y(y) = \int_0^{y+3} 1/8 dx = (y+3)/8$.

2a. No, X and Y are not independent, because they are not defined in rectangular regions.

2b. For $0 \leq x \leq 3$, we have $f_X(x) = \int_0^{3-x} 2/9 dy = 2/3 - 2x/9$.

2c. By symmetry, for $0 \leq y \leq 3$, we have $f_Y(y) = 2/3 - 2y/9$.

3a. Yes, X and Y are independent, and indeed, the joint density can be factored, as in **3b** and **3c**, so that $f_{X,Y}(x,y) = f_X(x)f_Y(y)$.

3b. We have $\int_0^\infty e^{-3x} dx = 1/3$, so $f_X(x) = 3e^{-3x}$ for $x > 0$, and $f_X(x) = 0$ otherwise.

3c. We have $\int_0^\infty e^{-2y} dy = 1/2$, so $f_Y(y) = 2e^{-2y}$ for $y > 0$, and $f_Y(y) = 0$ otherwise.

4a. Yes, X and Y are independent, and indeed, the joint density can be factored, as in **4b** and **4c**, so that $f_{X,Y}(x,y) = f_X(x)f_Y(y)$.

4b. For $-2 < x < 2$, we have $f_X(x) = \int_{-2}^2 \frac{9}{256} x^2 y^2 dy = \frac{3}{16} x^2$, and $f_X(x) = 0$ otherwise.

4c. By symmetry, for $-2 \leq y \leq 2$, we have $f_Y(y) = \frac{3}{16} y^2$, and $f_Y(y) = 0$ otherwise.

5a. *Method #1:* We can compute: $\mathbb{E}(XY) = \int_0^4 \int_0^3 (xy) \frac{1}{36} (4-x)(3-y) dy dx = 4/3$.

Method #2: Since X and Y are independent, we know $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$. We also have (from the previous problem set) $f_X(x) = \frac{1}{8}(4-x)$ for $0 < x < 4$, and $f_X(x) = 0$ otherwise, so $\mathbb{E}(X) = \int_0^4 (x) \frac{1}{8} (4-x) dx = 4/3$, and $f_Y(y) = \frac{2}{9}(3-y)$ for $0 < y < 3$, and $f_Y(y) = 0$ otherwise, so $\mathbb{E}(Y) = \int_0^3 (y) \frac{2}{9} (3-y) dy = 1$, so $\mathbb{E}(XY) = (4/3)(1) = 4/3$.

5b. *Method #1:* We can compute: $\mathbb{E}(X-Y) = \int_0^4 \int_0^3 (x-y) \frac{1}{36} (4-x)(3-y) dy dx = 1/3$.

Method #2: We have $\mathbb{E}(X-Y) = \mathbb{E}(X) - \mathbb{E}(Y)$. In Method #2 above, we saw $\mathbb{E}(X) = 4/3$ and $\mathbb{E}(Y) = 1$, so $\mathbb{E}(X-Y) = 4/3 - 1 = 1/3$.

6a. We compute $P(W > a) = P(X > a \text{ and } Y > a) = P(X > a)P(Y > a) = (\int_a^\infty 3e^{-3x} dx)(\int_a^\infty 2e^{-2y} dy) = e^{-3a}e^{-2a} = e^{-5a}$.

Therefore, $F_W(a) = P(W \leq a) = 1 - P(W > a) = 1 - e^{-5a}$. Thus, $F_W(w) = 1 - e^{-5w}$. Differentiating with respect to w , this yields $f_W(w) = 5e^{-5w}$.

6b. We have $\mathbb{E}(W) = \int_0^\infty (w)(5e^{-5w}) dw = 1/5$.