STAT/MA 41600
In-Class Problem Set #27: October 22, 2014
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1a. We have $f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$. The numerator must be 1/8 since the joint density is constant on a region with area 8. So the denominator is $f_Y(y) = \int_0^{y+3} 1/8\,dx = (y+3)/8$. Thus $f_{X|Y}(x \mid y) = \frac{1/8}{(y+3)/8} = 1/(y+3)$ for $0 < y < 2$, and $f_{X|Y}(x \mid y) = 0$ otherwise.
1b. We have $f_{X|Y}(x \mid 1) = 1/(1+3) = 1/4$. Thus $P(X \le 3 \mid Y = 1) = \int_0^3 f_{X|Y}(x \mid 1)\,dx = \int_0^3 1/4\,dx = 3/4$.
1c. We have $P(X \le 3 \mid Y \le 1) = \frac{P(X \le 3\ \&\ Y \le 1)}{P(Y \le 1)} = \frac{\int_0^1 \int_0^3 1/8\ dx\ dy}{\int_0^1 \int_0^{y+3} 1/8\ dx\ dy} = \frac{3/8}{7/16} = 6/7$.

2a. For $0 \le x \le 3$, we have $f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$. The numerator is again 1/8. The denominator is $f_X(x) = \int_0^2 1/8\,dy = 1/4$. Thus $f_{Y|X}(y \mid x) = \frac{1/8}{1/4} = 1/2$ for $0 < y < 2$, and $f_{Y|X}(y \mid x) = 0$ otherwise.
2b. For $3 \le x \le 5$, we have $f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$. The numerator is again 1/8. The denominator is $f_X(x) = \int_{x-3}^2 1/8\,dy = (5-x)/8$. Thus $f_{Y|X}(y \mid x) = \frac{1/8}{(5-x)/8} = 1/(5-x)$ for $x - 3 < y < 2$, and $f_{Y|X}(y \mid x) = 0$ otherwise.
2c. We have $f_{Y|X}(y \mid 2) = 1/2$. Thus $P(Y \le 1.5 \mid X = 2) = \int_0^{1.5} f_{Y|X}(y \mid 2)\ dy = \int_0^{1.5} 1/2\,dy = 3/4$.

3a. We have $f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$. The numerator is $10e^{-3x-2y}$. The denominator is $f_X(x) = \int_x^\infty 10e^{-3x-2y}\,dy = 5e^{-5x}$. Thus $f_{Y|X}(y \mid x) = \frac{10e^{-3x-2y}}{5e^{-5x}} = 2e^{2x-2y}$ for $x < y < \infty$, and $f_{Y|X}(y \mid x) = 0$ otherwise.
3b. We have $f_{Y|X}(y \mid 2) = 2e^{2(2)-2y} = 2e^{4-2y}$. Thus $P(Y > 3 \mid X = 2) = \int_3^\infty f_{Y|X}(y \mid 2)\,dy = \int_3^\infty 2e^{4-2y}\,dy = e^{-2}$.
3c. We have $P(Y > 3 \mid X > 2) = \frac{P(Y>3\ \&\ X>2)}{P(X>2)} = \frac{\int_3^\infty \int_2^y 10e^{-3x-2y}\ dx\ dy}{\int_2^\infty \int_2^y 10e^{-3x-2y}\ dx\ dy} = \frac{\frac{5}{3}e^{-12} - \frac{2}{3}e^{-15}}{e^{-10}} = \frac{5}{3}e^{-2} - \frac{2}{3}e^{-5} = 0.2211$.

4a. We have $f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$. The numerator is $10e^{-3x-2y}$. The denominator is $f_Y(y) = \int_0^y 10e^{-3x-2y}\,dx = \frac{10}{3}(e^{-2y} - e^{-5y})$. Thus $f_{X|Y}(x \mid y) = \frac{10e^{-3x-2y}}{\frac{10}{3}(e^{-2y}-e^{-5y})} = \frac{3e^{-3x-2y}}{e^{-2y}-e^{-5y}}$ for $0 < x < y$, and $f_{X|Y}(x \mid y) = 0$ otherwise.
4b. We have $f_{X|Y}(x \mid 3) = \frac{3e^{-3x-2(3)}}{e^{-2(3)}-e^{-5(3)}} = \frac{3e^{-3x-6}}{e^{-6}-e^{-15}}$. Thus $P(X > 2 \mid Y = 3) = \int_2^3 f_{X|Y}(x \mid 3)dx = \int_2^3 \frac{3e^{-3x-6}}{e^{-6}-e^{-15}}\,dx = \frac{e^{-12}-e^{-15}}{e^{-6}-e^{-15}} = \frac{e^3-1}{e^9-1} = 0.0023556$.
4c. We have $P(X > 2 \mid Y > 3) = \frac{P(X>2\ \&\ Y>3)}{P(Y>3)} = \frac{\int_3^\infty \int_2^y 10e^{-3x-2y}\ dx\ dy}{\int_3^\infty \int_0^y 10e^{-3x-2y}\ dx\ dy} = \frac{\frac{5}{3}e^{-12} - \frac{2}{3}e^{-15}}{\frac{5}{3}e^{-6} - \frac{2}{3}e^{-15}} = \frac{5e^3-2}{5e^9-2} = 0.0024295$.

5a. We have $f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$. The numerator is $\frac{1}{2}(2-x)(2-y)$. The denominator is $f_X(x) = \int_x^2 \frac{1}{2}(2-x)(2-y)\,dy = (1/4)(2-x)^3$. Thus $f_{Y|X}(y \mid x) = \frac{\frac{1}{2}(2-x)(2-y)}{(1/4)(2-x)^3} = \frac{2(2-y)}{(2-x)^2}$ for $x < y < 2$, and $f_{Y|X}(y \mid x) = 0$ otherwise.
5b. We have $f_{Y|X}(y \mid 1/2) = \frac{2(2-y)}{(2-1/2)^2} = \frac{8(2-y)}{9}$. Thus $P(Y > 1 \mid X = 1/2) = \int_1^2 f_{Y|X}(y \mid 1/2)dy = \int_1^2 \frac{8(2-y)}{9}\,dy = 4/9$.
5c. We have $P(Y > 1 \mid X > 1/2) = \frac{P(Y>1 \ \& \ X>1/2)}{P(X>1/2)} = \frac{\int_1^2 \int_{1/2}^y \frac{1}{2}(2-x)(2-y)\,dx\,dy}{\int_{1/2}^2 \int_{1/2}^y \frac{1}{2}(2-x)(2-y)\,dx\,dy} = \frac{7/32}{81/256} = \frac{56}{81}$.

6a. We have $f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$. The numerator is $\frac{1}{2}(2-x)(2-y)$. The denominator is $f_Y(y) = \int_0^y \frac{1}{2}(2-x)(2-y)\,dx = \frac{1}{4}y(2-y)(4-y)$. Thus $f_{X|Y}(x \mid y) = \frac{\frac{1}{2}(2-x)(2-y)}{\frac{1}{4}y(2-y)(4-y)} = \frac{2(2-x)}{y(4-y)}$ for $0 < x < y$, and $f_{X|Y}(x \mid y) = 0$ otherwise.
6b. We have $f_{X|Y}(x \mid 1) = \frac{2(2-x)}{1(4-1)} = (2/3)(2-x)$. Thus $P(X > 1/2 \mid Y = 1) = \int_{1/2}^1 f_{X|Y}(x \mid 1)\,dx = \int_{1/2}^1 (2/3)(2-x)\,dx = 5/12$.
6c. We have $P(X > 1/2 \mid Y > 1) = \frac{P(X>1/2 \ \& \ Y>1)}{P(Y>1)} = \frac{\int_1^2 \int_{1/2}^y \frac{1}{2}(2-x)(2-y)\,dx\,dy}{\int_1^2 \int_0^y \frac{1}{2}(2-x)(2-y)\,dx\,dy} = \frac{7/32}{7/16} = 1/2$.