

STAT/MA 41600
In-Class Problem Set #27: October 22, 2014
Solutions by Mark Daniel Ward

1a. We have $f_{X|Y}(x | y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$. The numerator must be $1/8$ since the joint density is constant on a region with area 8. So the denominator is $f_Y(y) = \int_0^{y+3} 1/8 dx = (y+3)/8$. Thus $f_{X|Y}(x | y) = \frac{1/8}{(y+3)/8} = 1/(y+3)$ for $0 < y < 2$, and $f_{X|Y}(x | y) = 0$ otherwise.

1b. We have $f_{X|Y}(x | 1) = 1/(1+3) = 1/4$. Thus $P(X \leq 3 | Y = 1) = \int_0^3 f_{X|Y}(x | 1) dx = \int_0^3 1/4 dx = 3/4$.

1c. We have $P(X \leq 3 | Y \leq 1) = \frac{P(X \leq 3 \ \& \ Y \leq 1)}{P(Y \leq 1)} = \frac{\int_0^1 \int_0^3 1/8 dx dy}{\int_0^1 \int_0^{y+3} 1/8 dx dy} = \frac{3/8}{7/16} = 6/7$.

2a. For $0 \leq x \leq 3$, we have $f_{Y|X}(y | x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$. The numerator is again $1/8$. The denominator is $f_X(x) = \int_0^2 1/8 dy = 1/4$. Thus $f_{Y|X}(y | x) = \frac{1/8}{1/4} = 1/2$ for $0 < y < 2$, and $f_{Y|X}(y | x) = 0$ otherwise.

2b. For $3 \leq x \leq 5$, we have $f_{Y|X}(y | x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$. The numerator is again $1/8$. The denominator is $f_X(x) = \int_{x-3}^2 1/8 dy = (5-x)/8$. Thus $f_{Y|X}(y | x) = \frac{1/8}{(5-x)/8} = 1/(5-x)$ for $x-3 < y < 2$, and $f_{Y|X}(y | x) = 0$ otherwise.

2c. We have $f_{Y|X}(y | 2) = 1/2$. Thus $P(Y \leq 1.5 | X = 2) = \int_0^{1.5} f_{Y|X}(y | 2) dy = \int_0^{1.5} 1/2 dy = 3/4$.

3a. We have $f_{Y|X}(y | x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$. The numerator is $10e^{-3x-2y}$. The denominator is $f_X(x) = \int_x^\infty 10e^{-3x-2y} dy = 5e^{-5x}$. Thus $f_{Y|X}(y | x) = \frac{10e^{-3x-2y}}{5e^{-5x}} = 2e^{2x-2y}$ for $x < y < \infty$, and $f_{Y|X}(y | x) = 0$ otherwise.

3b. We have $f_{Y|X}(y | 2) = 2e^{2(2)-2y} = 2e^{4-2y}$. Thus $P(Y > 3 | X = 2) = \int_3^\infty f_{Y|X}(y | 2) dy = \int_3^\infty 2e^{4-2y} dy = e^{-2}$.

3c. We have $P(Y > 3 | X > 2) = \frac{P(Y > 3 \ \& \ X > 2)}{P(X > 2)} = \frac{\int_2^\infty \int_3^y 10e^{-3x-2y} dx dy}{\int_2^\infty \int_2^y 10e^{-3x-2y} dx dy} = \frac{\frac{5}{3}e^{-12} - \frac{2}{3}e^{-15}}{e^{-10}} = \frac{5}{3}e^{-2} - \frac{2}{3}e^{-5} = 0.2211$.

4a. We have $f_{X|Y}(x | y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$. The numerator is $10e^{-3x-2y}$. The denominator is $f_Y(y) = \int_0^y 10e^{-3x-2y} dx = \frac{10}{3}(e^{-2y} - e^{-5y})$. Thus $f_{X|Y}(x | y) = \frac{10e^{-3x-2y}}{\frac{10}{3}(e^{-2y} - e^{-5y})} = \frac{3e^{-3x-2y}}{e^{-2y} - e^{-5y}}$ for $0 < x < y$, and $f_{X|Y}(x | y) = 0$ otherwise.

4b. We have $f_{X|Y}(x | 3) = \frac{3e^{-3x-2(3)}}{e^{-2(3)} - e^{-5(3)}} = \frac{3e^{-3x-6}}{e^{-6} - e^{-15}}$. Thus $P(X > 2 | Y = 3) = \int_2^3 f_{X|Y}(x | 3) dx = \int_2^3 \frac{3e^{-3x-6}}{e^{-6} - e^{-15}} dx = \frac{e^{-12} - e^{-15}}{e^{-6} - e^{-15}} = \frac{e^3 - 1}{e^9 - 1} = 0.0023556$.

4c. We have $P(X > 2 | Y > 3) = \frac{P(X > 2 \ \& \ Y > 3)}{P(Y > 3)} = \frac{\int_3^\infty \int_2^y 10e^{-3x-2y} dx dy}{\int_3^\infty \int_0^y 10e^{-3x-2y} dx dy} = \frac{\frac{5}{3}e^{-12} - \frac{2}{3}e^{-15}}{\frac{5}{3}e^{-6} - \frac{2}{3}e^{-15}} = \frac{5e^3 - 2}{5e^9 - 2} = 0.0024295$.

5a. We have $f_{Y|X}(y | x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$. The numerator is $\frac{1}{2}(2-x)(2-y)$. The denominator is $f_X(x) = \int_x^2 \frac{1}{2}(2-x)(2-y) dy = (1/4)(2-x)^3$. Thus $f_{Y|X}(y | x) = \frac{\frac{1}{2}(2-x)(2-y)}{(1/4)(2-x)^3} = \frac{2(2-y)}{(2-x)^2}$ for $x < y < 2$, and $f_{Y|X}(y | x) = 0$ otherwise.

5b. We have $f_{Y|X}(y | 1/2) = \frac{2(2-y)}{(2-1/2)^2} = \frac{8(2-y)}{9}$. Thus $P(Y > 1 | X = 1/2) = \int_1^2 f_{Y|X}(y | 1/2) dy = \int_1^2 \frac{8(2-y)}{9} dy = 4/9$.

5c. We have $P(Y > 1 | X > 1/2) = \frac{P(Y>1 \& X>1/2)}{P(X>1/2)} = \frac{\int_{1/2}^2 \int_{1/2}^y \frac{1}{2}(2-x)(2-y) dx dy}{\int_{1/2}^2 \int_{1/2}^y \frac{1}{2}(2-x)(2-y) dx dy} = \frac{7/32}{81/256} = \frac{56}{81}$.

6a. We have $f_{X|Y}(x | y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$. The numerator is $\frac{1}{2}(2-x)(2-y)$. The denominator is $f_Y(y) = \int_0^y \frac{1}{2}(2-x)(2-y) dx = \frac{1}{4}y(2-y)(4-y)$. Thus $f_{X|Y}(x | y) = \frac{\frac{1}{2}(2-x)(2-y)}{\frac{1}{4}y(2-y)(4-y)} = \frac{2(2-x)}{y(4-y)}$ for $0 < x < y$, and $f_{X|Y}(x | y) = 0$ otherwise.

6b. We have $f_{X|Y}(x | 1) = \frac{2(2-x)}{1(4-1)} = (2/3)(2-x)$. Thus $P(X > 1/2 | Y = 1) = \int_{1/2}^1 f_{X|Y}(x | 1) dx = \int_{1/2}^1 (2/3)(2-x) dx = 5/12$.

6c. We have $P(X > 1/2 | Y > 1) = \frac{P(X>1/2 \& Y>1)}{P(Y>1)} = \frac{\int_1^2 \int_{1/2}^y \frac{1}{2}(2-x)(2-y) dx dy}{\int_1^2 \int_0^y \frac{1}{2}(2-x)(2-y) dx dy} = \frac{7/32}{7/16} = 1/2$.