STAT/MA 41600
In-Class Problem Set #32: October 31, 2014
Solutions by Mark Daniel Ward

1a. We have $P(V < W) = \int_0^\infty \int_v^\infty 15e^{-3v-5w}\,dw\,dv = 3/8$.
1b. We have $P(\max(V,W) \le 1/10) = P(V \le 1/10 \;\&\; W \le 1/10) = P(V \le 1/10)P(W \le 1/10) = (1 - e^{-(3)(1/10)})(1 - e^{-(5)(1/10)}) = 0.1020$.

2a. We have $P(X > 1/4) = P(\min(V,W) > 1/4) = P(V > 1/4 \;\&\; W > 1/4) = P(V > 1/4)P(W > 1/4) = e^{-(3)(1/4)}e^{-(5)(1/4)} = e^{-2} = 0.1353$. Alternatively, since X is the minimum of independent exponential random variables, then X is exponential with $\lambda = 3 + 5 = 8$, so $P(X > 1/4) = e^{-(8)(1/4)} = e^{-2}$.
2b. We have $1/2 = P(X > a) = e^{-8a}$, so $\ln(1/2) = -8a$, and thus the median is $a = \frac{\ln(1/2)}{-8} = (1/8)\ln(2) = 0.0866$.

3a. We have $P(X < Y) = \int_0^\infty \int_x^\infty 14e^{-2x-7y}\,dy\,dx = 2/9$.
3b. Since X and Y are independent exponential random variables, then their minimum is an exponential random variable too, with $\lambda = 2 + 7 = 9$, so $P(\min(X,Y) \le 1/10) = 1 - e^{-(9)(1/10)} = 0.5934$.

4a. For $a > 0$, we have $F_U(a) = P(U \le a) = P(\max(X,Y) \le a) = P(X \le a,\ Y \le a) = P(X \le a)P(Y \le a) = (1 - e^{-2a})(1 - e^{-7a})$. For $a \le 0$, of course, $F_U(a) = 0$.
4b. For $u > 0$, we have $f_U(u) = \frac{d}{du}F_U(u) = \frac{d}{du}((1-e^{-2u})(1-e^{-7u})) = 2e^{-2u} + 7e^{-7u} - 9e^{-9u}$; for $u \le 0$, we have $f_U(u) = 0$.

5a. A constant, positive multiple of an exponential random variable is an exponential random variable too. Thus X is exponential, with $\mathbb{E}(X) = \mathbb{E}(3Y) = 3\mathbb{E}(Y) = 3/2$, so X has parameter $\lambda = 2/3$. Thus $P(X > 1) = \int_1^\infty (2/3)e^{-(2/3)x}\,dx = e^{-2/3} = 0.5134$.
5b. We have $1/2 = P(X > a) = e^{-(2/3)a}$, so $\ln(1/2) = -(2/3)a$, and thus the median is $a = \frac{\ln(1/2)}{-(2/3)} = (3/2)\ln(2) = 1.0397$.

6a. We have $P(1/2 < X < 3/2) = \int_{1/2}^{3/2} (2/3)e^{-(2/3)x}\,dx = e^{-1/3} - e^{-1} = 0.3487$.
6b. Since X is an exponential random variable with $\lambda = 2/3$, then $\mathrm{Var}\,(X) = 1/\lambda^2 = 9/4$.