STAT/MA 41600
In-Class Problem Set #34: November 7, 2014
Solutions by Mark Daniel Ward

1a. The CDF is $F_X(x) = 0$ for $x \leq 0$ and $F_X(x) = 1$ for $x \geq 1$. For $0 < a < 1$, the CDF is $F_X(a) = \int_0^a \frac{(3+10-1)!}{(3-1)!(10-1)!} x^{3-1}(1-x)^{10-1}\,dx = \int_{1-a}^1 660(1-u)^2u^9\,du = 1 - 660(1-a)^{10}(1/10 - (2/11)(1-a) + (1/12)(1-a)^2)$.
1b. The probability that X is less than $1/2$ is $F_X(1/2) = 1 - 660(1-1/2)^{10}(1/10-(2/11)(1-1/2) + (1/12)(1-1/2)^2) = 4017/4096$.

2a. We have $\int_0^1 f_X(x)\,dx = \int_0^1 \frac{(3+10-1)!}{(3-1)!(10-1)!} x^{3-1}(1-x)^{10-1}\,dx = \int_0^1 660(1-u)^2u^9\,du = 660(u^{10}/10 - 2u^{11}/11 + u^{12}/12)|_{u=0}^1 = 1$.
2b. We have $\mathbb{E}(X) = \int_0^1 x\,f_X(x)\,dx = \int_0^1 x \frac{(3+10-1)!}{(3-1)!(10-1)!} x^{3-1}(1-x)^{10-1}\,dx = \int_0^1 660x^3(1-x)^9\,dx = \int_0^1 660(1-u)^3u^9\,du = 660(u^{10}/10 - 3u^{11}/11 + 3u^{12}/12 - u^{13}/13)|_{u=0}^1 = 3/13$.

3a. The probability that X exceeds $1/3$ is $\int_{1/3}^1 \frac{(2+2-1)!}{(2-1)!(2-1)!} x^{2-1}(1-x)^{2-1}\,dx = \int_{1/3}^1 6x(1-x)\,dx = 6(x^2/2 - x^3/3)|_{x=1/3}^1 = 1 - 7/27 = 20/27$.
3b. We have $P(1/4 < X < 3/4) = \int_{1/4}^{3/4} \frac{(2+2-1)!}{(2-1)!(2-1)!} x^{2-1}(1-x)^{2-1}\,dx = \int_{1/4}^{3/4} 6x(1-x)\,dx = 6(x^2/2 - x^3/3)|_{x=1/4}^{3/4} = 27/32 - 5/32 = 11/16$.

4a. We have $\mathbb{E}(X) = \int_0^1 x\,f_X(x)\,dx = \int_0^1 x \frac{(2+2-1)!}{(2-1)!(2-1)!} x^{2-1}(1-x)^{2-1}\,dx = \int_0^1 6x^2(1-x)\,dx = 6(x^3/3 - x^4/4)|_{x=0}^1 = 1/2$.
4b. We have $\mathbb{E}(X^2) = \int_0^1 x^2\,f_X(x)\,dx = \int_0^1 x^2 \frac{(2+2-1)!}{(2-1)!(2-1)!} x^{2-1}(1-x)^{2-1}\,dx = \int_0^1 6x^3(1-x)\,dx = 6(x^4/4 - x^5/5)|_{x=0}^1 = 3/10$.
4c. Thus, the variance of X is $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 3/10 - (1/2)^2 = 1/20$.

5. The probability X exceeds U is $P(X > U) = \int_0^1 \int_u^\infty (1)(3e^{-3x})\,dx\,du = (1/3)(1 - e^{-3})$.

6. The probability X_1 is larger than X_2 is $P(X_1 > X_2) = \int_0^\infty \int_{x_2}^\infty \lambda_1 e^{-\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2}\,dx_1\,dx_2 = \lambda_2/(\lambda_1 + \lambda_2)$.