STAT/MA 41600
In-Class Problem Set #37: November 14, 2014
Solutions by Mark Daniel Ward

1a. The sum $X_1 + \cdots + X_{300}$ is a Gamma random variable with $\lambda = 5$ and $r = 300$.

1b. The expected value of $X_1 + \cdots + X_{300}$ is $r/\lambda = 300/5 = 60$. The variance is $r/\lambda^2 = 300/5^2 = 12$. The standard deviation is $\sqrt{r/\lambda^2} = \sqrt{12}$.

1c. An approximation is $P(58 < X_1+\cdots+X_{300} < 62) = P(\frac{58-60}{\sqrt{12}} < \frac{X_1+\cdots+X_{300}-60}{\sqrt{12}} < \frac{62-60}{\sqrt{12}}) \approx P(-.58 < Z < .58) = P(Z < .58) - (1 - P(Z < .58)) = 0.7190 - (1 - 0.7190) = 0.4380$.

1d. We estimate $P(Y < X_1+\cdots+X_{300}) = P(Y-(X_1+\cdots+X_{300}) < 0) = P(\frac{Y-(X_1+\cdots+X_{300})-(63-60)}{\sqrt{10+12}} < \frac{0-(63-60)}{\sqrt{10+12}}) \approx P(Z < -0.64) = P(Z > 0.64) = 1 - P(Z < 0.64) = 1 - 0.7389 = .2611$.

2. We have $\mathbb{E}(X) = r_X/\lambda_X = 200/2 = 100$ and $\mathrm{Var}(X) = r_X/\lambda_X^2 = 200/2^2 = 50$. Also $\mathbb{E}(Y) = r_Y/\lambda_Y = 312/3 = 104$ and $\mathrm{Var}(Y) = r_Y\lambda_Y^2 = 104/3$. Thus $P(X < Y) = P(0 < Y - X) = P(\frac{0-(104-100)}{\sqrt{50+104/3}} < \frac{Y-X-(104-100)}{\sqrt{50+104/3}}) \approx P(-0.43 < Z) = P(0.43 > Z) = 0.6664$.

3a. A good approximation is $P(140 < U_1+\cdots+U_{50} < 160) = P(\frac{140-50(3)}{\sqrt{50(3)}} < \frac{U_1+\cdots+U_{50}-50(3)}{\sqrt{50(3)}} < \frac{160-50(3)}{\sqrt{50(3)}}) \approx P(-0.82 < Z < 0.82) = P(Z < 0.82) - P(Z < -0.82) = P(Z < 0.82) - (1 - P(Z < 0.82)) = 0.7939 - (1 - 0.7939) = 0.5878$.

3b. A good approximation is $P(|U_1+\cdots+U_{50}-150| < 5) = P(145 < U_1+\cdots+U_{50} < 150) = P(\frac{145-50(3)}{\sqrt{50(3)}} < \frac{U_1+\cdots+U_{50}-50(3)}{\sqrt{50(3)}} < \frac{155-50(3)}{\sqrt{50(3)}}) \approx P(-0.41 < Z < 0.41) = P(Z < 0.41) - P(Z < -0.41) = P(Z < 0.41) - (1 - P(Z < 0.41)) = 0.6591 - (1 - 0.6591) = 0.3182$.

4a. The sum $X_1 + \cdots + X_{500}$ is a Negative Binomial random variable with $p = 5/8$ and $r = 500$.

4b. The expected value of $X_1 + \cdots + X_{500}$ is $r/p = 500/(5/8) = 800$. The variance is $qr/p^2 = (3/8)(500)/(5/8)^2 = 480$. The standard deviation is $\sqrt{qr/p^2} = \sqrt{480}$.

4c. An approximation—*using continuity correction*, since we are approximating a discrete random variable with a continuous random variable—is $P(780 < X_1 + \cdots + X_{500} < 820) = P(780.5 < X_1 + \cdots + X_{500} < 819.5) = P(\frac{780.5-800}{\sqrt{480}} < \frac{X_1+\cdots+X_{500}-800}{\sqrt{480}} < \frac{819.5-800}{\sqrt{480}}) \approx P(-.89 < Z < .89) = P(Z < .89) - (1 - P(Z < .89)) = 0.8133 - (1 - 0.8133) = 0.6266$.

4d. We have $\mathbb{E}(Y) = r/p = 250/(1/3) = 750$ and $\mathrm{Var}(Y) = qr/p^2 = (2/3)(250)/(1/3)^2 = 1500$. An approximation—*using continuity correction*, since we are approximating a discrete random variable with a continuous random variable—is $P(Y < X_1 + \cdots + X_{500}) = P(Y - (X_1 + \cdots + X_{500}) < 0) = P(Y - (X_1 + \cdots + X_{500}) < -0.5) = P(\frac{Y-(X_1+\cdots+X_{500})-(750-800)}{\sqrt{480+1500}} < \frac{-0.5-(750-800)}{\sqrt{480+1500}}) \approx P(Z < 1.11) = .8665$.

5. An approximation—*using continuity correction*, since we are approximating a discrete random variable with a continuous random variable—is $P(290 < X < 310) = P(290.5 < X < 309.5) = P(\frac{290.5-300}{\sqrt{300}} < \frac{X-300}{\sqrt{300}} < \frac{309.5-300}{\sqrt{300}}) \approx P(-.55 < Z < .55) = P(Z < .55) - P(Z < -.55) = P(Z < .55) - (1 - P(Z < .55)) = 0.7088 - (1 - 0.7088) = 0.4176$.

6a. We have $\mathbb{E}(X_j) = \int_0^6 x(x^2/72)\,dx = 9/2$.

6b. We have $\mathbb{E}(X_j^2) = \int_0^6 x^2(x^2/72)\,dx = 21.6$. Thus $\mathrm{Var}(X) = 21.6 - (9/2)^2 = 27/20$.

6c. We have $P(X_1 + \cdots + X_{100} < 460) = P(\frac{X_1+\cdots+X_{100}-100(9/2)}{\sqrt{100(27/20)}} < \frac{460-100(9/2)}{\sqrt{100(27/20)}}) \approx P(Z < 0.86) = .8051$.