1a. The masses of $X$ and $Y$ are:

\[ f_X(1) = 11/36; \quad f_X(2) = 9/36; \quad f_X(3) = 7/36; \quad f_X(4) = 5/36; \quad f_X(5) = 3/36; \quad f_X(6) = 1/36; \]
\[ f_Y(1) = 1/36; \quad f_Y(2) = 3/36; \quad f_Y(3) = 5/36; \quad f_Y(4) = 7/36; \quad f_Y(5) = 9/36; \quad f_Y(6) = 11/36; \]

Thus, we compute $E(X) = 91/36$ and $E(Y) = 161/36$.

We compute $E(XY) = 49/4$ in the same way as in #5 of Practice Problem Set #39. So $\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = 49/4 - (91/36)(161/36) = 1225/1296$.

1b. Using the masses of $X$ and $Y$, we compute $E(X^2) = 301/36$ and $E(Y^2) = 791/36$, so $\text{Var}(X) = E(X^2) - (E(X))^2 = 301/36 - (91/36)^2 = 2555/1296$ and $\text{Var}(Y) = E(Y^2) - (E(Y))^2 = 791/36 - (161/36)^2 = 2555/1296$. So the correlation of $X$ and $Y$ is $\rho_{X,Y} = \text{Cov}(X,Y)/\sqrt{\text{Var}(X)\text{Var}(Y)} = (1225/1296)/\sqrt{(2555/1296)(2555/1296)} = 357/373$.

2. We let $X_j = 1$ if the $j$th person gets their own CD back, and $X_j = 0$ otherwise, so $X = X_1 + \cdots + X_6$. So $\text{Var}(X) = \text{Var}(X_1 + \cdots + X_6) = \sum_{i=1}^6 \text{Var}(X_i) + \sum_{i \neq j} \text{Cov}(X_i, X_j)$. We have $\text{Var}(X_i) = E(X_i^2) - (E(X_i))^2 = 1/6 - (1/6)^2 = (1/6)(5/6)$ and $\text{Cov}(X_i, X_j) = E(X_iX_j) - E(X_i)E(X_j) = (1/6)(1/5) - (1/6)(1/6)$. So $\text{Var}(X) = (6)(1/6)(5/6) + (30)(1/6)(1/5) - (1/6)(1/6)) = 1$.

3a. We let $X_j = 1$ if the $j$th card is a Heart, and $X_j = 0$ otherwise, so $X = X_1 + \cdots + X_5$. We compute $E(X) = E(X_1 + \cdots + X_5) = E(X_1) + \cdots + E(X_5) = 13/52 + \cdots + 13/52 = 5/4$ and $\text{Var}(X) = \text{Var}(X_1 + \cdots + X_5) = \sum_{i=1}^5 \text{Var}(X_i) + \sum_{i \neq j} \text{Cov}(X_i, X_j)$. We have $\text{Var}(X_i) = E(X_i^2) - (E(X_i))^2 = 13/52 - (13/52)^2 = (1/4)(3/4)$ and $\text{Cov}(X_i, X_j) = E(X_iX_j) - E(X_i)E(X_j) = (13/52)(12/51) - (13/52)(13/52)$. So $\text{Var}(X) = (5)(1/4)(3/4) + (20)(13/52)(12/51) - (13/52)(13/52)) = 235/272$.

3b. We let $Y_j = 1$ if the $j$th card is a Queen, and $Y_j = 0$ otherwise, so $Y = Y_1 + \cdots + Y_5$. We compute $E(Y) = E(Y_1 + \cdots + Y_5) = E(Y_1) + \cdots + E(Y_5) = 4/52 + \cdots + 4/52 = 5/13$ and $\text{Var}(Y) = \text{Var}(Y_1 + \cdots + Y_5) = \sum_{i=1}^5 \text{Var}(Y_i) + \sum_{i \neq j} \text{Cov}(Y_i, Y_j)$. We have $\text{Var}(Y_i) = E(Y_i^2) - (E(Y_i))^2 = 4/52 - (4/52)^2 = (1/13)(12/13)$ and $\text{Cov}(Y_i, Y_j) = E(Y_iY_j) - E(Y_i)E(Y_j) = (4/52)(3/51) - (4/52)(4/52)$. So $\text{Var}(Y) = (5)(1/13)(12/13) + (20)(4/52)(3/51) - (4/52)(4/52) = 940/2873$.

3c. We have $X_iY_j = 1$ if the $i$th card is the Queen of Hearts; otherwise, $X_iY_j = 0$, so $E(X_iY_j) = 1/52$. Also, for $i \neq j$, we have $X_iY_j = 1$ if the $i$th card is a non-Queen and a heart, and the $j$th card is a Queen (which happens with probability $(12/52)(4/51)$) OR if the $i$th card is the Queen of Hearts and the $j$th card is a Queen (which happens with probability $(1/52)(3/51)$); otherwise, $X_iY_j = 0$; so we conclude $E(X_iY_j) = (12/52)(4/51) + (1/52)(3/51) = 1/52$. Now we proceed to solve the problem in two different ways.

Method #1: We first note that $\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$. We then compute $E(XY) = E((X_1 + \cdots + X_5)(Y_1 + \cdots + Y_5)) = \sum_{i=1}^5 E(X_iY_i) + \sum_{i \neq j} E(X_iY_j) = (5)(1/52) + (20)(1/52) = 25/52$. Thus $\text{Cov}(X,Y) = 25/52 - (5/4)(5/13) = 0$.

Method #2: We can (alternatively) compute $\text{Cov}(X,Y) = \text{Cov}(X_1 + \cdots + X_5, Y_1 + \cdots + Y_5) = \sum_{i=1}^5 \text{Cov}(X_i, Y_i) + \sum_{i \neq j} \text{Cov}(X_i, Y_j)$. We have $\text{Cov}(X_i, Y_i) = E(X_iY_i) - E(X_i)E(Y_i) = 1/52 - (1/4)(1/13) = 0$, and $\text{Cov}(X_i, Y_j) = 1/52 - (1/4)(1/13) = 0$. So altogether $\text{Cov}(X,Y) = \sum_{i=1}^5 \text{Cov}(X_i, Y_i) + \sum_{i \neq j} \text{Cov}(X_i, Y_j) = (5)(0) + (20)(0) = 0$. 

1
4a. We have $E(X) = \int_0^1 \int_0^{y+3}(x)(1/8) \, dx \, dy = 49/24$ and $E(Y) = \int_0^2 \int_0^{y+3}(y)(1/8) \, dx \, dy = 13/12$ and $E(XY) = \int_0^2 \int_0^{y+3}(xy)(1/8) \, dx \, dy = 19/8$. Thus, we conclude $Cov(X, Y) = E(XY) - E(X)E(Y) = 19/8 - (49/24)(13/12) = 47/288.$

4b. We have $E(X^2) = \int_0^2 \int_0^{y+3}(x^2)(1/8) \, dx \, dy = 17/3$ and $E(Y^2) = \int_0^2 \int_0^{y+3}(y^2)(1/8) \, dx \, dy = 3/2$. Thus $Var(X) = E(X^2) - (E(X))^2 = 17/3 - (49/24)^2 = 863/576$ and $Var(Y) = E(Y^2) - (E(Y))^2 = 3/2 - (13/12)^2 = 47/144$. We conclude that $\rho_{X,Y} = Cov(X, Y)/\sqrt{Var(X)Var(Y)} = (47/288)/\sqrt{(863/576)(47/144)} = (47/288)/\sqrt{40561/288} = 47/\sqrt{40561} = 47/863\sqrt{40561}.$

5a. We have $E(X) = \int_0^1 \int_0^{3-y}(x)(2/9) \, dx \, dy = 1$ and $E(Y) = \int_0^1 \int_0^{3-y}(y)(2/9) \, dx \, dy = 1$ and $E(XY) = \int_0^1 \int_0^{3-y}(xy)(2/9) \, dx \, dy = 3/4$. Thus, we conclude $Cov(X, Y) = E(XY) - E(X)E(Y) = 3/4 - (1)(1) = -1/4$.

5b. We have $E(X^2) = \int_0^3 \int_0^{3-y}(x^2)(2/9) \, dx \, dy = 3/2$ and $E(Y^2) = \int_0^3 \int_0^{3-y}(y^2)(2/9) \, dx \, dy = 3/2$. Thus $Var(X) = E(X^2) - (E(X))^2 = 3/2 - (1)^2 = 1/2$ and $Var(Y) = E(Y^2) - (E(Y))^2 = 3/2 - (1)^2 = 1/2$. We conclude that $\rho_{X,Y} = Cov(X, Y)/\sqrt{Var(X)Var(Y)} = (-1/4)/\sqrt{(1/2)(1/2)} = (-1/4)/(1/2) = -1/2$.

6a. We have $E(Y) = \int_0^\infty \int_0^y(60e^{-4y^2}) \, dx \, dy = 4/15$, or equivalently, we compute $E(Y) = \int_0^\infty \int_0^\infty (60e^{-4x^2-6y}) \, dy \, dx = 4/15$.

6b. We have $E(XY) = \int_0^\infty \int_0^y(xy)(60e^{-4x^2-6y}) \, dx \, dy = 11/300$, or equivalently, we compute $E(XY) = \int_0^\infty \int_0^\infty (xy)(60e^{-4x^2-6y}) \, dy \, dx = 11/300$.

6c. We conclude that $Cov(X, Y) = E(XY) - E(X)E(Y) = 11/300 - (1/10)(4/15) = 1/100$. 