

**1a.** The masses of  $X$  and  $Y$  are:

$$f_X(1) = 11/36; f_X(2) = 9/36; f_X(3) = 7/36; f_X(4) = 5/36; f_X(5) = 3/36; f_X(6) = 1/36; \\ f_Y(1) = 1/36; f_Y(2) = 3/36; f_Y(3) = 5/36; f_Y(4) = 7/36; f_Y(5) = 9/36; f_Y(6) = 11/36;$$

Thus, we compute  $\mathbb{E}(X) = 91/36$  and  $\mathbb{E}(Y) = 161/36$ .

We compute  $\mathbb{E}(XY) = 49/4$  in the same way as in #5 of Practice Problem Set #39. So  $\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 49/4 - (91/36)(161/36) = 1225/1296$ .

**1b.** Using the masses of  $X$  and  $Y$ , we compute  $\mathbb{E}(X^2) = 301/36$  and  $\mathbb{E}(Y^2) = 791/36$ , so  $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 301/36 - (91/36)^2 = 2555/1296$  and  $\text{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 791/36 - (161/36)^2 = 2555/1296$ . So the correlation of  $X$  and  $Y$  is  $\rho_{X,Y} = \text{Cov}(X, Y) / \sqrt{\text{Var}(X)\text{Var}(Y)} = (1225/1296) / \sqrt{(2555/1296)(2555/1296)} = 35/73$ .

**2.** We let  $X_j = 1$  if the  $j$ th person gets their own CD back, and  $X_j = 0$  otherwise, so  $X = X_1 + \dots + X_6$ . So  $\text{Var}(X) = \text{Var}(X_1 + \dots + X_6) = \sum_{i=1}^6 \text{Var}(X_i) + \sum_{i \neq j} \text{Cov}(X_i, X_j)$ . We have  $\text{Var}(X_i) = \mathbb{E}(X_i^2) - (\mathbb{E}(X_i))^2 = 1/6 - (1/6)^2 = (1/6)(5/6)$  and  $\text{Cov}(X_i, X_j) = \mathbb{E}(X_i X_j) - \mathbb{E}(X_i)\mathbb{E}(X_j) = (1/6)(1/5) - (1/6)(1/6)$ . So  $\text{Var}(X) = (6)(1/6)(5/6) + (30)((1/6)(1/5) - (1/6)(1/6)) = 1$ .

**3a.** We let  $X_j = 1$  if the  $j$ th card is a Heart, and  $X_j = 0$  otherwise, so  $X = X_1 + \dots + X_5$ . We compute  $\mathbb{E}(X) = \mathbb{E}(X_1 + \dots + X_5) = \mathbb{E}(X_1) + \dots + \mathbb{E}(X_5) = 13/52 + \dots + 13/52 = 5/4$  and  $\text{Var}(X) = \text{Var}(X_1 + \dots + X_5) = \sum_{i=1}^5 \text{Var}(X_i) + \sum_{i \neq j} \text{Cov}(X_i, X_j)$ . We have  $\text{Var}(X_i) = \mathbb{E}(X_i^2) - (\mathbb{E}(X_i))^2 = 13/52 - (13/52)^2 = (1/4)(3/4)$  and  $\text{Cov}(X_i, X_j) = \mathbb{E}(X_i X_j) - \mathbb{E}(X_i)\mathbb{E}(X_j) = (13/52)(12/51) - (13/52)(13/52)$ . So  $\text{Var}(X) = (5)(1/4)(3/4) + (20)((13/52)(12/51) - (13/52)(13/52)) = 235/272$ .

**3b.** We let  $Y_j = 1$  if the  $j$ th card is a Queen, and  $Y_j = 0$  otherwise, so  $Y = Y_1 + \dots + Y_5$ . We compute  $\mathbb{E}(Y) = \mathbb{E}(Y_1 + \dots + Y_5) = \mathbb{E}(Y_1) + \dots + \mathbb{E}(Y_5) = 4/52 + \dots + 4/52 = 5/13$  and  $\text{Var}(Y) = \text{Var}(Y_1 + \dots + Y_5) = \sum_{i=1}^5 \text{Var}(Y_i) + \sum_{i \neq j} \text{Cov}(Y_i, Y_j)$ . We have  $\text{Var}(Y_i) = \mathbb{E}(Y_i^2) - (\mathbb{E}(Y_i))^2 = 4/52 - (4/52)^2 = (1/13)(12/13)$  and  $\text{Cov}(Y_i, Y_j) = \mathbb{E}(Y_i Y_j) - \mathbb{E}(Y_i)\mathbb{E}(Y_j) = (4/52)(3/51) - (4/52)(4/52)$ . So  $\text{Var}(Y) = (5)(1/13)(12/13) + (20)((4/52)(3/51) - (4/52)(4/52)) = 940/2873$ .

**3c.** We have  $X_i Y_i = 1$  if the  $i$ th card is the Queen of Hearts; otherwise,  $X_i Y_i = 0$ , so  $\mathbb{E}(X_i Y_i) = 1/52$ . Also, for  $i \neq j$ , we have  $X_i Y_j = 1$  if the  $i$ th card is a non-Queen and a heart, and the  $j$ th card is a Queen (which happens with probability  $(12/52)(4/51)$ ) OR if the  $i$ th card is the Queen of Hearts and the  $j$ th card is a Queen (which happens with probability  $(1/52)(3/51)$ ); otherwise,  $X_i Y_j = 0$ ; so we conclude  $\mathbb{E}(X_i Y_j) = (12/52)(4/51) + (1/52)(3/51) = 1/52$ . Now we proceed to solve the problem in two different ways.

*Method #1:* We first note that  $\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$ . We then compute  $\mathbb{E}(XY) = \mathbb{E}((X_1 + \dots + X_5)(Y_1 + \dots + Y_5)) = \sum_{i=1}^5 \mathbb{E}(X_i Y_i) + \sum_{i \neq j} \mathbb{E}(X_i Y_j) = (5)(1/52) + (20)(1/52) = 25/52$ . Thus  $\text{Cov}(X, Y) = 25/52 - (5/4)(5/13) = 0$ .

*Method #2:* We can (alternatively) compute  $\text{Cov}(X, Y) = \text{Cov}(X_1 + \dots + X_5, Y_1 + \dots + Y_5) = \sum_{i=1}^5 \text{Cov}(X_i, Y_i) + \sum_{i \neq j} \text{Cov}(X_i, Y_j)$ . We have  $\text{Cov}(X_i, Y_i) = \mathbb{E}(X_i Y_i) - \mathbb{E}(X_i)\mathbb{E}(Y_i) = 1/52 - (1/4)(1/13) = 0$ , and  $\text{Cov}(X_i, Y_j) = \mathbb{E}(X_i Y_j) - \mathbb{E}(X_i)\mathbb{E}(Y_j) = 1/52 - (1/4)(1/13) = 0$ . So altogether  $\text{Cov}(X, Y) = \sum_{i=1}^5 \text{Cov}(X_i, Y_i) + \sum_{i \neq j} \text{Cov}(X_i, Y_j) = (5)(0) + (20)(0) = 0$ .

**4a.** We have  $\mathbb{E}(X) = \int_0^2 \int_0^{y+3} (x)(1/8) dx dy = 49/24$  and  $\mathbb{E}(Y) = \int_0^2 \int_0^{y+3} (y)(1/8) dx dy = 13/12$  and  $\mathbb{E}(XY) = \int_0^2 \int_0^{y+3} (xy)(1/8) dx dy = 19/8$ . Thus, we conclude  $\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 19/8 - (49/24)(13/12) = 47/288$ .

**4b.** We have  $\mathbb{E}(X^2) = \int_0^2 \int_0^{y+3} (x^2)(1/8) dx dy = 17/3$  and  $\mathbb{E}(Y^2) = \int_0^2 \int_0^{y+3} (y^2)(1/8) dx dy = 3/2$ . Thus  $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 17/3 - (49/24)^2 = 863/576$  and  $\text{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 3/2 - (13/12)^2 = 47/144$ . We conclude that  $\rho_{X,Y} = \text{Cov}(X, Y) / \sqrt{\text{Var}(X)\text{Var}(Y)} = (47/288) / \sqrt{(863/576)(47/144)} = (47/288) / \sqrt{40561/288^2} = 47 / \sqrt{40561} = \frac{1}{863} \sqrt{40561}$ .

**5a.** We have  $\mathbb{E}(X) = \int_0^3 \int_0^{3-y} (x)(2/9) dx dy = 1$  and  $\mathbb{E}(Y) = \int_0^3 \int_0^{3-y} (y)(2/9) dx dy = 1$  and  $\mathbb{E}(XY) = \int_0^3 \int_0^{3-y} (xy)(2/9) dx dy = 3/4$ . Thus, we conclude  $\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 3/4 - (1)(1) = -1/4$ .

**5b.** We have  $\mathbb{E}(X^2) = \int_0^3 \int_0^{3-y} (x^2)(2/9) dx dy = 3/2$  and  $\mathbb{E}(Y^2) = \int_0^3 \int_0^{3-y} (y^2)(2/9) dx dy = 3/2$ . Thus  $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 3/2 - (1)^2 = 1/2$  and  $\text{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 3/2 - (1)^2 = 1/2$ . We conclude that  $\rho_{X,Y} = \text{Cov}(X, Y) / \sqrt{\text{Var}(X)\text{Var}(Y)} = (-1/4) / \sqrt{(1/2)(1/2)} = (-1/4) / (1/2) = -1/2$ .

**6a.** We have  $\mathbb{E}(Y) = \int_0^\infty \int_0^y (y)(60e^{-4x-6y}) dx dy = 4/15$ , or equivalently, we compute  $\mathbb{E}(Y) = \int_0^\infty \int_x^\infty (y)(60e^{-4x-6y}) dy dx = 4/15$ .

**6b.** We have  $\mathbb{E}(XY) = \int_0^\infty \int_0^y (xy)(60e^{-4x-6y}) dx dy = 11/300$ , or equivalently, we compute  $\mathbb{E}(XY) = \int_0^\infty \int_x^\infty (xy)(60e^{-4x-6y}) dy dx = 11/300$ .

**6c.** We conclude that  $\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 11/300 - (1/10)(4/15) = 1/100$ .