STAT/MA 41600
In-Class Problem Set #39: November 24, 2014
Solutions by Mark Daniel Ward

1a. The masses of X and Y are:
$f_X(1) = 11/36$; $f_X(2) = 9/36$; $f_X(3) = 7/36$; $f_X(4) = 5/36$; $f_X(5) = 3/36$; $f_X(6) = 1/36$;
$f_Y(1) = 1/36$; $f_Y(2) = 3/36$; $f_Y(3) = 5/36$; $f_Y(4) = 7/36$; $f_Y(5) = 9/36$; $f_Y(6) = 11/36$;

Thus, we compute $\mathbb{E}(X) = 91/36$ and $\mathbb{E}(Y) = 161/36$.

We compute $\mathbb{E}(XY) = 49/4$ in the same way as in #5 of Practice Problem Set #39. So $\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 49/4 - (91/36)(161/36) = 1225/1296$.

1b. Using the masses of X and Y, we compute $\mathbb{E}(X^2) = 301/36$ and $\mathbb{E}(Y^2) = 791/36$, so $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 301/36 - (91/36)^2 = 2555/1296$ and $\mathrm{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 791/36 - (161/36)^2 = 2555/1296$. So the correlation of X and Y is $\rho_{X,Y} = \mathrm{Cov}(X,Y)/\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)} = (1225/1296)/\sqrt{(2555/1296)(2555/1296)} = 35/73$.

2. We let $X_j = 1$ if the jth person gets their own CD back, and $X_j = 0$ otherwise, so $X = X_1 + \cdots + X_6$. So $\mathrm{Var}(X) = \mathrm{Var}(X_1 + \cdots + X_6) = \sum_{i=1}^{6} \mathrm{Var}(X_i) + \sum_{i \neq j} \mathrm{Cov}(X_i, X_j)$. We have $\mathrm{Var}(X_i) = \mathbb{E}(X_i^2) - (\mathbb{E}(X_i))^2 = 1/6 - (1/6)^2 = (1/6)(5/6)$ and $\mathrm{Cov}(X_i, X_j) = \mathbb{E}(X_i X_j) - \mathbb{E}(X_i)\mathbb{E}(X_j) = (1/6)(1/5) - (1/6)(1/6)$. So $\mathrm{Var}(X) = (6)(1/6)(5/6) + (30)((1/6)(1/5) - (1/6)(1/6)) = 1$.

3a. We let $X_j = 1$ if the jth card is a Heart, and $X_j = 0$ otherwise, so $X = X_1 + \cdots + X_5$. We compute $\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_5) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_5) = 13/52 + \cdots + 13/52 = 5/4$ and $\mathrm{Var}(X) = \mathrm{Var}(X_1 + \cdots + X_5) = \sum_{i=1}^{5} \mathrm{Var}(X_i) + \sum_{i \neq j} \mathrm{Cov}(X_i, X_j)$. We have $\mathrm{Var}(X_i) = \mathbb{E}(X_i^2) - (\mathbb{E}(X_i))^2 = 13/52 - (13/52)^2 = (1/4)(3/4)$ and $\mathrm{Cov}(X_i, X_j) = \mathbb{E}(X_i X_j) - \mathbb{E}(X_i)\mathbb{E}(X_j) = (13/52)(12/51) - (13/52)(13/52)$. So $\mathrm{Var}(X) = (5)(1/4)(3/4) + (20)((13/52)(12/51) - (13/52)(13/52)) = 235/272$.

3b. We let $Y_j = 1$ if the jth card is a Queen, and $Y_j = 0$ otherwise, so $Y = Y_1 + \cdots + Y_5$. We compute $\mathbb{E}(Y) = \mathbb{E}(Y_1 + \cdots + Y_5) = \mathbb{E}(Y_1) + \cdots + \mathbb{E}(Y_5) = 4/52 + \cdots + 4/52 = 5/13$ and $\mathrm{Var}(Y) = \mathrm{Var}(Y_1 + \cdots + Y_5) = \sum_{i=1}^{5} \mathrm{Var}(Y_i) + \sum_{i \neq j} \mathrm{Cov}(Y_i, Y_j)$. We have $\mathrm{Var}(Y_i) = \mathbb{E}(Y_i^2) - (\mathbb{E}(Y_i))^2 = 4/52 - (4/52)^2 = (1/13)(12/13)$ and $\mathrm{Cov}(Y_i, Y_j) = \mathbb{E}(Y_i Y_j) - \mathbb{E}(Y_i)\mathbb{E}(Y_j) = (4/52)(3/51) - (4/52)(4/52)$. So $\mathrm{Var}(Y) = (5)(1/13)(12/13) + (20)((4/52)(3/51) - (4/52)(4/52)) = 940/2873$.

3c. We have $X_i Y_i = 1$ if the ith card is the Queen of Hearts; otherwise, $X_i Y_i = 0$, so $\mathbb{E}(X_i Y_i) = 1/52$. Also, for $i \neq j$, we have $X_i Y_j = 1$ if the ith card is a non-Queen and a heart, and the jth card is a Queen (which happens with probability $(12/52)(4/51)$) OR if the ith card is the Queen of Hearts and the jth card is a Queen (which happens with probability $(1/52)(3/51)$); otherwise, $X_i Y_j = 0$; so we conclude $\mathbb{E}(X_i Y_j) = (12/52)(4/51) + (1/52)(3/51) = 1/52$. Now we proceed to solve the problem in two different ways.

Method #1: We first note that $\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$. We then compute $\mathbb{E}(XY) = \mathbb{E}((X_1 + \cdots + X_5)(Y_1 + \cdots + Y_5)) = \sum_{i=1}^{5} \mathbb{E}(X_i Y_i) + \sum_{i \neq j} \mathbb{E}(X_i Y_j) = (5)(1/52) + (20)(1/52) = 25/52$. Thus $\mathrm{Cov}(X,Y) = 25/52 - (5/4)(5/13) = 0$.

Method #2: We can (alternatively) compute $\mathrm{Cov}(X,Y) = \mathrm{Cov}(X_1 + \cdots + X_5, Y_1 + \cdots + Y_5) = \sum_{i=1}^{5} \mathrm{Cov}(X_i, Y_i) + \sum_{i \neq j} \mathrm{Cov}(X_i, Y_j)$. We have $\mathrm{Cov}(X_i, Y_i) = \mathbb{E}(X_i Y_i) - \mathbb{E}(X_i)\mathbb{E}(Y_i) = 1/52 - (1/4)(1/13) = 0$, and $\mathrm{Cov}(X_i, Y_j) = 1/52 - (1/4)(1/13) = 0$. So altogether $\mathrm{Cov}(X,Y) = \sum_{i=1}^{5} \mathrm{Cov}(X_i, Y_i) + \sum_{i \neq j} \mathrm{Cov}(X_i, Y_j) = (5)(0) + (20)(0) = 0$.

4a. We have $\mathbb{E}(X) = \int_0^2 \int_0^{y+3} (x)(1/8)\, dx\, dy = 49/24$ and $\mathbb{E}(Y) = \int_0^2 \int_0^{y+3} (y)(1/8)\, dx\, dy = 13/12$ and $\mathbb{E}(XY) = \int_0^2 \int_0^{y+3} (xy)(1/8)\, dx\, dy = 19/8$. Thus, we conclude $\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 19/8 - (49/24)(13/12) = 47/288$.
4b. We have $\mathbb{E}(X^2) = \int_0^2 \int_0^{y+3} (x^2)(1/8)\, dx\, dy = 17/3$ and $\mathbb{E}(Y^2) = \int_0^2 \int_0^{y+3} (y^2)(1/8)\, dx\, dy = 3/2$. Thus $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 17/3 - (49/24)^2 = 863/576$ and $\mathrm{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 3/2 - (13/12)^2 = 47/144$. We conclude that $\rho_{X,Y} = \mathrm{Cov}(X,Y)/\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)} = (47/288)/\sqrt{(863/576)(47/144)} = (47/288)/\sqrt{40561/288^2} = 47/\sqrt{40561} = \frac{1}{863}\sqrt{40561}$.

5a. We have $\mathbb{E}(X) = \int_0^3 \int_0^{3-y} (x)(2/9)\, dx\, dy = 1$ and $\mathbb{E}(Y) = \int_0^3 \int_0^{3-y} (y)(2/9)\, dx\, dy = 1$ and $\mathbb{E}(XY) = \int_0^3 \int_0^{3-y} (xy)(2/9)\, dx\, dy = 3/4$. Thus, we conclude $\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 3/4 - (1)(1) = -1/4$.
5b. We have $\mathbb{E}(X^2) = \int_0^3 \int_0^{3-y} (x^2)(2/9)\, dx\, dy = 3/2$ and $\mathbb{E}(Y^2) = \int_0^3 \int_0^{3-y} (y^2)(2/9)\, dx\, dy = 3/2$. Thus $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 3/2 - (1)^2 = 1/2$ and $\mathrm{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 3/2 - (1)^2 = 1/2$. We conclude that $\rho_{X,Y} = \mathrm{Cov}(X,Y)/\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)} = (-1/4)/\sqrt{(1/2)(1/2)} = (-1/4)/(1/2) = -1/2$.

6a. We have $\mathbb{E}(Y) = \int_0^\infty \int_0^y (y)(60e^{-4x-6y})\, dx\, dy = 4/15$, or equivalently, we compute $\mathbb{E}(Y) = \int_0^\infty \int_x^\infty (y)(60e^{-4x-6y})\, dy\, dx = 4/15$.
6b. We have $\mathbb{E}(XY) = \int_0^\infty \int_0^y (xy)(60e^{-4x-6y})\, dx\, dy = 11/300$, or equivalently, we compute $\mathbb{E}(XY) = \int_0^\infty \int_x^\infty (xy)(60e^{-4x-6y})\, dy\, dx = 11/300$.
6c. We conclude that $\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 11/300 - (1/10)(4/15) = 1/100$.