STAT/MA 41600
In-Class Problem Set #40: December 1, 2014
Solutions by Mark Daniel Ward

1a. The conditional mass of X, given $Y = 5$, is $f_{X|Y}(5 \mid 5) = 1/3$; $f_{X|Y}(6 \mid 5) = 2/3$; and $f_{X|Y}(x \mid 5) = 0$ otherwise. So $\mathbb{E}(X \mid Y = 5) = (5)(1/3) + (6)(2/3) = 17/3$.

1b. The conditional mass of Y, given $X = 5$, is $f_{Y|X}(1 \mid 5) = 2/9$; $f_{Y|X}(2 \mid 5) = 2/9$; $f_{Y|X}(3 \mid 5) = 2/9$; $f_{Y|X}(4 \mid 5) = 2/9$; $f_{Y|X}(5 \mid 5) = 1/9$; $f_{Y|X}(y \mid 5) = 0$ otherwise. So $\mathbb{E}(Y \mid X = 5) = (1)(2/9) + (2)(2/9) + (3)(2/9) + (4)(2/9) + (5)(1/9) = 25/9$.

2a. The mass of X is $p_X(x) = \sum_{y=x}^{\infty}(2/3)^x(1/2)^y = (1/3)^{x-1}(2/3)$ for $x \geq 1$, and $p_X(x) = 0$ otherwise.

2b. The conditional mass of Y, given $X = x$, is $p_{Y|X}(y \mid x) = \frac{p_{X,Y}(x,y)}{p_X(x)} = \frac{(2/3)^x(1/2)^y}{(1/3)^{x-1}(2/3)} = (1/2)^{y-x+1}$ for $y \geq x$, and $p_{Y|X}(y \mid x) = 0$ otherwise.

2c. The conditional mass $p_{Y|X}(y \mid x)$ is nonnegative in all cases. Also, $\sum_{y=x}^{\infty} p_{Y|X}(y \mid x) = \sum_{y=x}^{\infty}(1/2)^{y-x+1} = (1/2)\sum_{y=x}^{\infty}(1/2)^{y-x} = (1/2)\sum_{y=0}^{\infty}(1/2)^y = \frac{1/2}{1-1/2} = 1$. So $p_{Y|X}(y \mid x)$ is a valid probability mass function.

2d. We compute $\mathbb{E}(Y \mid X = x) = \sum_{y=x}^{\infty} y p_{Y|X}(y \mid x) = \sum_{y=x}^{\infty} y(1/2)^{y-x+1} = (1/2)\sum_{y=x}^{\infty} y(1/2)^{y-x}$. Then we shift the y's by x, to get $\mathbb{E}(Y \mid X = x) = (1/2)\sum_{y=0}^{\infty}(y + x)(1/2)^y$. Then we can split into two terms, so $\mathbb{E}(Y \mid X = x) = (1/2)\sum_{y=0}^{\infty}(y)(1/2)^y + (1/2)\sum_{y=0}^{\infty}(x)(1/2)^y = (1/2)\sum_{y=0}^{\infty}(y)(1/2)^{y-1}(1/2) + \frac{(1/2)(x)}{1-1/2}$. The second term is just x. The first term is 1/2 times the expected value of a geometric random variable with parameter 1/2, which is 2, so the first term is 1 altogether. Thus $\mathbb{E}(Y \mid X = x) = 1 + x$.

3a. Let $X_k = 1$ if the kth card is a Heart, and $X_k = 0$ otherwise. Then $X = X_1 + \cdots + X_Y$. For fixed y, we have $\mathbb{E}(X \mid Y = y) = \mathbb{E}(X_1 + \cdots + X_Y \mid Y = y) = \mathbb{E}(X_1 + \cdots + X_y \mid Y = y) = \mathbb{E}(X_1 + \cdots + X_y) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_y) = 13/52 + \cdots + 13/52 = (y)(13/52) = y/4$.

3b. We have $\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X \mid Y)) = \mathbb{E}(Y(13/52)) = (13/52)\mathbb{E}(Y) = (13/52)(10) = 5/2$.

Alternatively, i.e., if you prefer this way, we have $\mathbb{E}(X) = \sum_y \mathbb{E}(X \mid Y = y)P(Y = y) = \sum_y y(13/52)P(Y = y) = (13/52)\sum_y y\,P(Y = y) = (13/52)\mathbb{E}(Y) = (13/52)(10) = 5/2$.

4a. The density of Y, for $0 \leq y \leq 2$, if $f_Y(y) = \int_0^{y+3} f_{X,Y}(x,y)\,dx = \int_0^{y+3}(1/8)\,dx = (y+3)/8$, and $f_Y(y) = 0$ otherwise.

4b. The conditional density of X, given $Y = y$, is $f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1/8}{(y+3)/8} = 1/(y+3)$ for $0 \leq x \leq y + 3$, and $f_{X|Y}(x \mid y) = 0$ otherwise.

4c. The conditional density $f_{X|Y}(x \mid y)$ is nonnegative in all cases. Also, $\int_0^{y+3} f_{X|Y}(x \mid y)\,dx = \int_0^{y+3} 1/(y + 3)\,dx = 1$. So $f_{X|Y}(x \mid y)$ is a valid probability density function.

4d. We compute $\mathbb{E}(X \mid Y = y) = \int_0^{y+3}(x)(1/(y + 3))\,dx = (y + 3)/2$.

5a. We have $p_{Y|X}(1 \mid 1) = 3/51$ and $p_{Y|X}(0 \mid 1) = 48/51$, so $\mathbb{E}(Y \mid X = 1) = (1)(3/51) + (0)(48/51) = 3/51$.

5b. We have $p_{Y|X}(1 \mid 0) = 4/51$ and $p_{Y|X}(0 \mid 0) = 47/51$, so $\mathbb{E}(Y \mid X = 0) = (1)(4/51) + (0)(47/51) = 4/51$.

5c. Switching the role of X and Y in **5a**, we have, by symmetry, $\mathbb{E}(X \mid Y = 1) = 3/51$.

5d. Switching the role of X and Y in **5b**, we have, by symmetry, $\mathbb{E}(X \mid Y = 0) = 4/51$.

6a. We have $f_X(x) = \int_x^\infty 60e^{-4x-6y}\,dy = 10e^{-10x}$. Thus $f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{60e^{-4x-6y}}{10e^{-10x}} = 6e^{-6(y-x)}$ for $y > x$, and $f_{Y|X}(y \mid x) = 0$ otherwise. Thus $\mathbb{E}(Y \mid X = x) = \int_x^\infty (y)(6e^{-6(y-x)})\,dy = x + 1/6$.

6b. We have $f_Y(y) = \int_0^y 60e^{-4x-6y}\,dx = 15e^{-6y} - 15e^{-10y}$. Thus $f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{60e^{-4x-6y}}{15e^{-6y}-15e^{-10y}} = \frac{4e^{-4x}}{1-e^{-4y}}$ for $0 < x < y$, and $f_{X|Y}(x \mid y) = 0$ otherwise. Thus $\mathbb{E}(X \mid Y = y) = \int_0^y (x)\frac{4e^{-4x}}{1-e^{-4y}}\,dx = \frac{1}{4}\frac{e^{4y}-4y-1}{e^{4y}-1} = \frac{1}{4} - \frac{y}{e^{4y}-1}$.