STAT/MA 41600
In-Class Problem Set #42: December 5, 2014

1. Consider three independent Exponential random variables X_1, X_2, X_3, each with mean 1.
1a. Find the density of $X_{(1)} = \min(X_1, X_2, X_3)$.
1b. Compute $\mathbb{E}(X_{(1)})$.
1c. Find the density of the second order statistic, $X_{(2)}$, i.e., the second-smallest one.
1d. Compute $\mathbb{E}(X_{(2)})$.

2. Same setup as in **1**.
2a. Find the density of $X_{(3)} = \max(X_1, X_2, X_3)$.
2b. Compute $\mathbb{E}(X_{(3)})$.
2c. Sanity check: We know that $X_1 + X_2 + X_3 = X_{(1)} + X_{(2)} + X_{(3)}$. Therefore, we have $\mathbb{E}(X_{(1)}) + \mathbb{E}(X_{(2)}) + \mathbb{E}(X_{(3)}) = \mathbb{E}(X_{(1)} + X_{(2)} + X_{(3)}) = \mathbb{E}(X_1 + X_2 + X_3) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) = 1 + 1 + 1 = 3$. So please make sure your answers to **1b**, **1d**, and **2b** sum to 3 too.

3. Consider a circle of radius 3. Let X_1 and X_2 be two points, each chosen Uniformly at random in the circle. Let W_1 and W_2 be their respective distances to the origin.
3a. Find the CDF of W_1. (By symmetry, W_2 has the same CDF.)
3b. Find the density of W_1. (By symmetry, W_2 has the same density too.)
3c. Find the expected value of W_1. (By symmetry, W_2 has the same expected value too.)

4. Same setup as in **3**. Let $W_{(1)}$ and $W_{(2)}$ be the order statistics of the pair W_1, W_2.
4a. Find the density of $W_{(1)} = \min(W_1, W_2)$.
4b. Compute $\mathbb{E}(W_{(1)})$.
4c. Find the density of $W_{(2)} = \max(W_1, W_2)$.
4d. Compute $\mathbb{E}(W_{(2)})$.
4e. Sanity check: We know that $W_1 + W_2 = W_{(1)} + W_{(2)}$. So $\mathbb{E}(W_{(1)}) + \mathbb{E}(W_{(2)}) = \mathbb{E}(W_{(1)} + W_{(2)}) = \mathbb{E}(W_1 + W_2) = \mathbb{E}(W_1) + \mathbb{E}(W_2)$. So please make sure that your answers to **4b** and **4d** have the same sum as we would find if we compared to $\mathbb{E}(W_1) + \mathbb{E}(W_2)$ from **3c**.

5. Let U_1, U_2, U_3, U_4, U_5 be five independent, continuous random variables, each uniformly distributed on $[0, 10]$. Then $U_{(4)}$ denotes the second-largest of these five random variables.
5a. Find the probability density function of $U_{(4)}$.
5b. Find the mean of $U_{(4)}$.

6. (Review question) Suppose that heights of blades of grass have expected value 4 inches and standard deviation 0.75 inches. (Do not assume that the heights are Normally distributed.)
6a. Find a bound on the probability that a randomly selected blade of grass is at least 6.5 inches tall.
6b. A recent commercial says that this kind of grass has a *good looking appearance* when it is between 2.75 to 5.25 inches tall. Find a bound on the probability that a randomly selected blade of grass has this type of *good looking appearance*.