STAT/MA 41600
In-Class Problem Set #42: December 5, 2014
Solutions by Mark Daniel Ward

1a. Using the density $f_X(x) = e^{-x}$ and CDF $F_X(x) = 1 - e^{-x}$, the general formula for the density of order statistics tells us $f_{X_{(1)}}(x) = \binom{3}{0,1,2}e^{-x}(1-e^{-x})^0(e^{-x})^2 = 3e^{-3x}$. Thus, $X_{(1)}$ is an exponential random variable with parameter $\lambda = 3$.

1b. We compute $\mathbb{E}(X_{(1)}) = \int_0^\infty (x)(3e^{-3x})\,dx = 1/3$.

1c. As before, the general formula for the density of order statistics tells us $f_{X_{(2)}}(x) = \binom{3}{1,1,1}e^{-x}(1-e^{-x})^1(e^{-x})^1 = 6e^{-2x}(1-e^{-x})$

1d. We compute $\mathbb{E}(X_{(2)}) = \int_0^\infty (x)(6e^{-2x}(1-e^{-x}))\,dx = \int_0^\infty (x)(6e^{-2x})\,dx - \int_0^\infty (x)(6e^{-3x})\,dx = 3\int_0^\infty (x)(2e^{-2x})\,dx - 2\int_0^\infty (x)(3e^{-3x})\,dx = 3(1/2) - 2(1/3) = 5/6$.

2a. As before, the general formula for the density of order statistics tells us $f_{X_{(3)}}(x) = \binom{3}{2,1,0}e^{-x}(1-e^{-x})^2(e^{-x})^0 = 3e^{-x}(1-e^{-x})^2$.

2b. We compute $\mathbb{E}(X_{(3)}) = \int_0^\infty (x)(3e^{-x}(1-e^{-x})^2)\,dx = \int_0^\infty (x)(3e^{-x}(1-2e^{-x}+e^{-2x}))\,dx = \int_0^\infty (x)(3e^{-x})\,dx - \int_0^\infty (x)(6e^{-2x})\,dx + \int_0^\infty (x)(3e^{-3x})\,dx = 3\int_0^\infty (x)(e^{-x})\,dx - 3\int_0^\infty (x)(2e^{-2x})\,dx + \int_0^\infty (x)(3e^{-3x})\,dx = 3(1) - 3(1/2) + 1/3 = 11/6$.

2c. We check that we have $\mathbb{E}(X_{(1)}) + \mathbb{E}(X_{(2)}) + \mathbb{E}(X_{(3)}) = 1/3 + 5/6 + 11/6 = 3$, as desired.

3a. For $0 \le a \le 3$, we have $F_{W_1}(a) = P(W \le a) = (a^2\pi)/(9\pi) = a^2/9$. Thus $F_{W_1}(w) = w^2/9$ for $0 \le w \le 3$, and $F_{W_1}(w) = 0$ for $w < 0$, and $F_{W_1}(w) = 1$ for $w > 3$.

3b. For $0 \le w \le 3$, we have $f_{W_1}(w) = \frac{d}{dw}(w^2/9) = 2w/9$, and $f_{W_1}(w) = 0$ otherwise.

3c. We compute $\mathbb{E}(W_1) = \int_0^3 (w)(2w/9)\,dw = 2$.

4a. Using the CDF and density from question **3**, the general formula for the density of order statistics tells us $f_{W_{(1)}}(w) = \binom{2}{0,1,1}(2w/9)(w^2/9)^0(1-w^2/9)^1 = (4/9)w - (4/81)w^3$.

4b. We compute $\mathbb{E}(W_{(1)}) = \int_0^3 (w)((4/9)w - (4/81)w^3)\,dw = 8/5$.

4c. Again using the CDF and density from question **3**, the general formula for the density of order statistics tells us $f_{W_{(2)}}(w) = \binom{2}{1,1,0}(2w/9)(w^2/9)^1(1-w^2/9)^0 = (4/81)w^3$.

4d. We compute $\mathbb{E}(W_{(2)}) = \int_0^3 (w)((4/81)w^3)\,dw = 12/5$.

4e. We have $\mathbb{E}(W_1) + \mathbb{E}(W_2) = 2 + 2 = 4$. Similarly, $\mathbb{E}(W_{(1)}) + \mathbb{E}(W_{(2)}) = 8/5 + 12/5 = 4$, as desired.

5a. We have $f_{U_{(4)}}(u) = \binom{5}{3,1,1}(1/10)(u/10)^3(1-u/10)^1 = 2(u/10)^3 - 2(u/10)^4$ for $0 \le u \le 10$, and $f_{U_{(4)}}(u) = 0$ otherwise.

5b. We have $\mathbb{E}(U_{(4)}) = \int_0^{10} (u)(2(u/10)^3 - 2(u/10)^4)\,du = 20/3$.

6a. By Markov's inequality, if we let X denote the height, we have $P(X \ge 6.5) \le (4)/(6.5) = 8/13 = 0.6154$.

6b. By Chebyshev's inequality, we have $P(2.75 \le X \le 5.25) = P(|X-4| \le 1.25) = P(|X-4| \le (5/3)(0.75)) \ge 1 - 1/(5/3)^2 = 16/25 = 0.64$.