

STAT/MA 41600
In-Class Problem Set #44: December 10, 2014
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1a. We compute $\mathbb{E}(X^2) = \int_0^3 (x^2)(1/3) dx = 3$.

1b. We compute $\mathbb{E}((X^2)^2) = \int_0^3 (x^4)(1/3) dx = 81/5$. Thus, we conclude $\text{Var}(X) = \mathbb{E}((X^2)^2) - (\mathbb{E}(X^2))^2 = 81/5 - 3^2 = 36/5$.

2a. The random variable Y can take on values 0 to 9.

2b. For $0 \leq a \leq 9$, we have $F_Y(a) = P(Y \leq a) = P(X^2 \leq a) = P(X \leq \sqrt{a}) = \frac{\sqrt{a}}{3}$. Therefore, we have $F_Y(y) = 0$ for $y < 0$, and $F_Y(y) = 1$ for $y > 9$, and $F_Y(y) = \frac{\sqrt{y}}{3}$.

2c. For $0 \leq y \leq 9$, we have $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} \frac{\sqrt{y}}{3} = \frac{1}{6\sqrt{y}}$, and $f_Y(y) = 0$ otherwise.

2d. We compute $\mathbb{E}(Y) = \int_0^9 (y)(\frac{1}{6\sqrt{y}}) dy = \int_0^9 (\frac{\sqrt{y}}{6}) dy = 3$.

2e. We compute $\mathbb{E}(Y^2) = \int_0^9 (y^2)(\frac{1}{6\sqrt{y}}) dy = 81/5$. Thus, we have $\text{Var}(Y) = 81/5 - 3^2 = 36/5$.

3a. We have $\mathbb{E}(3X + 2) = 3\mathbb{E}(X) + 2 = 3(1/4) + 2 = 11/4$.

3b. We have $\text{Var}(3X + 2) = 3^2 \text{Var}(X) = 3^2/4^2 = 9/16$.

4a. The random variable Y can take on values in $[2, \infty)$.

4b. For $a \geq 0$, we have $F_Y(a) = P(Y \leq a) = P(3X + 2 \leq a) = P(X \leq (a - 2)/3) = 1 - e^{-4(a-2)/3}$. So we have $F_Y(y) = 1 - e^{-4(a-2)/3}$ for $y \geq 0$, and $F_Y(y) = 0$ otherwise.

4c. For $y \geq 0$, we have $f_Y(y) = \frac{d}{dy}(1 - e^{-4(a-2)/3}) = (4/3)e^{-4(y-2)/3}$, and $f_Y(y) = 0$ otherwise.

4d. We have $\mathbb{E}(Y) = \int_2^\infty (y)(4/3)e^{-4(y-2)/3} dy = \int_0^\infty (y+2)(4/3)e^{-4y/3} dy = \int_0^\infty (y)(4/3)e^{-4y/3} dy + 2 \int_0^\infty (4/3)e^{-4y/3} dy = 3/4 + 2(1) = 11/4$.

4e. We have $\mathbb{E}(Y^2) = \int_2^\infty (y^2)(4/3)e^{-4(y-2)/3} dy = \int_0^\infty (y + 2)^2(4/3)e^{-4y/3} dy = \int_0^\infty (y^2 + 4y + 4)(4/3)e^{-4y/3} dy = \int_0^\infty (y^2)(4/3)e^{-4y/3} dy + 4 \int_0^\infty (y)(4/3)e^{-4y/3} dy + 4 \int_0^\infty (4/3)e^{-4y/3} dy = 2(3/4)^2 + 4(3/4) + 4 = 65/8$. Therefore, we conclude $\text{Var}(Y) = 65/8 - (11/4)^2 = 9/16$.