STAT/MA 41600
Midterm Exam 1 Answers
Friday, October 10, 2014
Solutions by Mark Daniel Ward

1. *Method #1:* Let B denote the event that "black" appears on top, and let A denote the event that the fair die (with 3 white sides and 3 black sides) was chosen. Then $P(A \mid B) = \frac{P(A\cap B)}{P(B)} = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A)+P(B \mid A^c)P(A^c)} = \frac{(1/2)(1/2)}{(1/2)(1/2)+(2/3)(1/2)} = 3/7$.

Method #2: There are seven black sides on the dice, and all are equally likely to appear, and three of them belong to the fair die. So the desired probability is 3/7.

2a. Let X_j indicate if the jth group consists of 1 Stat and 1 Math student, i.e., $X_j = 1$ if this happens, and $X_j = 0$ otherwise. Then $X = X_1 + \cdots + X_{20}$, so $\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_{20}) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_{20}) = 20/39 + \cdots + 20/39 = 20(20/39) = 400/39 = 10.2564$.

2b. We have $X^2 = (X_1+\cdots+X_{20})^2$, so $\mathbb{E}(X^2) = \mathbb{E}((X_1 + \cdots + X_{20})^2)$, and when we expand, we get 20 terms that each equal $\mathbb{E}(X_1)$, and $20^2 - 20 = 380$ terms that each equal $\mathbb{E}(X_1X_2)$. We already observed $\mathbb{E}(X_1) = 20/39$. Also $\mathbb{E}(X_1X_2) = P(X_1X_2 = 1) = (20/39)(19/37)$. Thus $\mathbb{E}(X^2) = (20)(20/39) + (380)(20/39)(19/37) = 159200/1443 = 110.3257$. So $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 288800/56277 = 5.1318$.

3. *Method #1:* We have $P(X \geq y) = \sum_{y=1}^{\infty}\sum_{x=y}^{\infty} p_{X,Y}(x,y) = \sum_{y=1}^{\infty}\sum_{x=y}^{\infty}(\frac{3}{4})^{x-1}(\frac{1}{4})(\frac{1}{2})^y = (\frac{1}{4})(\frac{1}{2})\sum_{y=1}^{\infty}(\frac{1}{2})^{y-1}\sum_{x=y}^{\infty}(\frac{3}{4})^{x-1}$. The inner sum is $\sum_{x=y}^{\infty}(\frac{3}{4})^{x-1} = \frac{(\frac{3}{4})^{y-1}}{1-\frac{3}{4}}$. Thus $P(X \geq Y) = (\frac{1}{4})(\frac{1}{2})\sum_{y=1}^{\infty}(\frac{1}{2})^{y-1}\frac{(\frac{3}{4})^{y-1}}{1-\frac{3}{4}} = (\frac{1}{2})\sum_{y=1}^{\infty}((\frac{1}{2})(\frac{3}{4}))^{y-1} = (\frac{1}{2})\sum_{y=1}^{\infty}(\frac{3}{8})^{y-1} = (\frac{1}{2})/(1-\frac{3}{8}) = \frac{4}{5}$.

Method #2: We want the probability that the first head occurs earlier, or at the same time, as the first "1." Of the 8 equally-likely results (x,y) with $1 \leq x \leq 4$ and $y = H$ or $y = T$, there are 5 equally-likely results in which $x = 1$ or $y = H$ or both, namely: $(1,T),(1,H),(2,H),(3,H),(4,H)$. Given that one of these 5 equally-likely results occurs, it is one of the 4 results with H (not T) with probability 4/5.

4a. The exact probability is $\sum_{x=0}^{4}\binom{300{,}000}{x}(\frac{1}{100{,}000})^x(\frac{99{,}999}{100{,}000})^{300{,}000-x}$, namely,

$$\binom{300{,}000}{0}\left(\frac{1}{100{,}000}\right)^0\left(\frac{99{,}999}{100{,}000}\right)^{300{,}000} + \binom{300{,}000}{1}\left(\frac{1}{100{,}000}\right)^1\left(\frac{99{,}999}{100{,}000}\right)^{299{,}999}$$
$$+ \binom{300{,}000}{2}\left(\frac{1}{100{,}000}\right)^2\left(\frac{99{,}999}{100{,}000}\right)^{299{,}998} + \binom{300{,}000}{3}\left(\frac{1}{100{,}000}\right)^3\left(\frac{99{,}999}{100{,}000}\right)^{299{,}997}$$
$$+ \binom{300{,}000}{4}\left(\frac{1}{100{,}000}\right)^4\left(\frac{99{,}999}{100{,}000}\right)^{299{,}996}$$

4b. If $n = 300{,}000$ and $p = 1/100{,}000$, then the probability in part **4a** is the probability that a Binomial n, p random variable is 4 or less. Now let Y be Poisson with average $\lambda = np = 3$. So the Poisson approximation is $P(X \leq 4) \approx P(Y \leq 4) = \frac{e^{-3}3^0}{0!} + \frac{e^{-3}3^1}{1!} + \frac{e^{-3}3^2}{2!} + \frac{e^{-3}3^3}{3!} + \frac{e^{-3}3^4}{4!}$.

4c. We have $P(X \leq 4) \approx P(Y \leq 4) = \frac{e^{-3}3^0}{0!} + \frac{e^{-3}3^1}{1!} + \frac{e^{-3}3^2}{2!} + \frac{e^{-3}3^3}{3!} + \frac{e^{-3}3^4}{4!} = 0.8153$.

Please see some notes on the reverse side of this sheet.

Notes.

Question 1 was the same as Problem Set #5, question 6, using dice instead of coins.

Question 2 was the same as Problem Set #20/#22, question 3, with some numbers changed and with the additional question about the variance. We had extensive practice with these kinds of variance questions, especially throughout Problem Set #12.

Question 3 was the same as Problem Set #16, question 6, with some numbers changed.

Question 4 was the same as in the (take-home) Practice Problems #19, question 5, with some numbers changed. We had practice with these kinds of Poisson approximations to Binomial random variables in Problem Set #18, especially questions 4 and 5.