STAT/MA 41600
Midterm Exam 2 Answers
Friday, November 21, 2014
Solutions by Mark Daniel Ward

1. *Method #1:* We have $P(Y > X) = \int_0^\infty \int_x^\infty 12e^{-3x-4y}\,dy\,dx = \int_0^\infty 3e^{-7x}\,dx = 3/7$.

Method #2: We have $P(Y > X) = \int_0^\infty \int_0^y 12e^{-3x-4y}\,dx\,dy = \int_0^\infty (4e^{-4y} - 4e^{-7y})\,dy = 3/7$.

2. *Method #1:* For $x > 0$, we have $f_X(x) = \int_x^\infty 60e^{-4x-6y}\,dy = 10e^{-10x}$. Thus X is an Exponential random variable with $\lambda = 10$ so $\mathbb{E}(X) = 1/10$.

Method #2: We have $\mathbb{E}(X) = \int_0^\infty \int_x^\infty (x)(60e^{-4x-6y})\,dy dx = \int_0^\infty (x)(60e^{-4x}) \int_x^\infty e^{-6y}\,dy dx = \int_0^\infty (x)(60e^{-4x})(\frac{1}{6}e^{-6x})\,dx = \int_0^\infty (x)(10e^{-10x})\,dx = 1/10$.

Method #3: We have $\mathbb{E}(X) = \int_0^\infty \int_0^y (x)(60e^{-4x-6y})\,dx\,dy = \int_0^\infty (60e^{-6y}) \int_0^y xe^{-4x}\,dx\,dy = \int_0^\infty (60e^{-6y})(\frac{1}{16} - \frac{1}{16}e^{-4y} - \frac{1}{4}ye^{-4y})\,dy = \int_0^\infty (\frac{15}{4}e^{-6y} - \frac{15}{4}e^{-10y} - 15ye^{-10y})\,dy = 1/10$.

3a. We have $F_X(x) = 0$ for $x < 0$ and $F_X(x) = 1$ for $x > 10$. For $0 \le a \le 10$, we have $F_X(a) = P(X \le a) = P(\max(U, V) \le a) = P(U \le a,\ V \le a) = P(U \le a)P(V \le a) = (a/10)(a/10) = a^2/100$. So we conclude $F_X(x) = x^2/100$ for $0 \le x \le 10$.

3b. We have $f_X(x) = F_X'(x)$. Thus, for $0 \le x \le 10$, we have $f_X(x) = \frac{d}{dx}(x^2/100) = x/50$; otherwise, we have $f_X(x) = 0$.

3c. To be a valid probability density function, the function must be nonnegative and must integrate to 1. We have $f_X(x) \ge 0$ for all x. Also, we have $\int_{-\infty}^\infty f_X(x)\,dx = \int_0^{10} x/50\,dx = 1$. So $f_X(x)$ is a valid probability density function.

4a. We have $\mathbb{E}(U_j) = \int_0^6 (u)(1/6)\,du = 3$.

4b. We have $\mathbb{E}(U_j^2) = \int_0^6 (u^2)(1/6)\,du = 12$.

4c. Using **4a** and **4b**, we get $\mathrm{Var}(U_j) = \mathbb{E}(U_j^2) - (\mathbb{E}(U_j))^2 = 12 - 3^2 = 3$.

4d. We have $P(U_1 + \cdots + U_{200} < 625) = P(\frac{U_1+\cdots+U_{200}-200(3)}{\sqrt{200(3)}} < \frac{625-200(3)}{\sqrt{200(3)}}) \approx P(Z < 1.02) = 0.8461$.

5. We note that X and Y are each Poisson random variables with a large parameter, so $X - Y$ has an approximately Normal distribution. Of course, $X - Y$ is a discrete random variable (it is integer valued), so we must use continuity correction. Thus, we compute $P(Y - X \ge 20) = P(Y - X \ge 19.5) = P(\frac{Y-X-(450-450)}{\sqrt{450+450}} \ge \frac{19.5-(450-450)}{\sqrt{450+450}}) \approx P(Z \ge 0.65) = 1 - P(Z \le 0.65) = 1 - 0.7422 = 0.2578$.

Please see some notes on the reverse side of this sheet.

Notes.

Question 1 was the same as Problem Set #25, question 3a (only one of three parts on this problem), with the numbers changed.

Question 2 was the same as Problem Set #28, question 3, with the numbers changed.

Question 3 was the same as Problem Set #30/31, question 3, with only two variables instead of three variables.

Question 4 was a simplified version of Problem Set #37, question 3. The midterm exam version was even easier, since we only had one inequality, not two inequalities.

Question 5 was a simplified version of Problem Set #37, part 2, question 3. The midterm exam version was even easier, since we removed the absolute value on the midterm exam.