Example of Bayes’ Theorem (third version).

Suppose we put five different dice into a hat. The dice have the following number of sides: 4, 6, 8, 12, 20. When we choose a die from the hat, each of the five of the dice are equally likely to appear.

Suppose that a “3” appears. What is the probability it was the 4-sided die that was chosen?

Let \( A_1 \) be the event that the 4-sided die was chosen, \( A_2 \) be the event that the 6-sided die was chosen, \( A_3 \) be the event that the 8-sided die was chosen, \( A_4 \) be the event that the 12-sided die was chosen, and \( A_5 \) be the event that the 20-sided die was chosen. Notice that \( A_j \)’s are disjoint (non-overlapping) and that the union of the \( A_j \)’s is all of \( S \). Let \( B \) denote the event that a “3” appears on the chosen die. Notice we don’t know \( P(B) \) either! We use this form of Bayes’ Theorem:

\[
P(A_j \mid B) = \frac{P(A_j)P(B \mid A_j)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + P(A_3)P(B \mid A_3) + P(A_4)P(B \mid A_4) + P(A_5)P(B \mid A_5)}
\]

In particular, we focus on \( j = 1 \) case.

\[
P(A_1 \mid B) = \frac{(1/5)(1/4)}{(1/5)(1/4) + (1/5)(1/6) + (1/5)(1/8) + (1/5)(1/12) + (1/5)(1/20)} = 10/27 = 0.37
\]

The probability that the 6-sided die was chosen, given that “3” appeared, is

\[
P(A_2 \mid B) = \frac{(1/5)(1/6)}{(1/5)(1/4) + (1/5)(1/6) + (1/5)(1/8) + (1/5)(1/12) + (1/5)(1/20)} = 20/81 = 0.25
\]

The probability that the 8-sided die was chosen, given that “3” appeared, is

\[
P(A_3 \mid B) = \frac{(1/5)(1/8)}{(1/5)(1/4) + (1/5)(1/6) + (1/5)(1/8) + (1/5)(1/12) + (1/5)(1/20)} = 5/27 = 0.19
\]

The probability that the 12-sided die was chosen, given that “3” appeared, is

\[
P(A_4 \mid B) = \frac{(1/5)(1/12)}{(1/5)(1/4) + (1/5)(1/6) + (1/5)(1/8) + (1/5)(1/12) + (1/5)(1/20)} = 10/81 = 0.12
\]

The probability that the 20-sided die was chosen, given that “3” appeared, is

\[
P(A_5 \mid B) = \frac{(1/5)(1/20)}{(1/5)(1/4) + (1/5)(1/6) + (1/5)(1/8) + (1/5)(1/12) + (1/5)(1/20)} = 2/27 = 0.07
\]

Finally, we notice that \( P(A_1|B) + P(A_2|B) + P(A_3|B) + P(A_4|B) + P(A_5|B) = 1 \), as we know it should.