Three equivalent views of the statement that X and Y are independent random variables. You can treat any one of them as the definition, and show that the other two are equivalent to the chosen statement.

1. The product of the masses of X and Y gives the joint mass of X and Y, in other words

$$p_{X,Y}(x,y) = p_X(x)p_Y(y)$$

In other words,

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$

We must have this hold for all x, y, to have independence between X and Y.

2. The CDF's of the X and Y multiply together to give the joint CDF of X and Y, i.e.,

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)$$

Or in terms of probabilities,

$$P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y)$$

Again, we need this to hold for all pairs x, y to have independence of the random variables X and Y.

3. The third view: Use conditional masses. If the mass of X at x equals the conditional mass of X given $Y = y$, for all x, y with the $P(Y = y) > 0$, then X and Y are independent. In other words,

$$p_X(x) = P_{X|Y}(x|y)$$

for all x, y with $p_Y(y) > 0$. Similarly, another condition (variant of condition 3) is to switch the roles of X and Y in the statement of 3. This is like a statement 3b, i.e., a 4th equivalently version of independence.