

Expected values of sums of random variables

Remember we had two ways (equivalent) to define expected values. The first way, very common, is to say  $E(X) = \sum_j x_j P(X = x_j)$ . We see this in a lot of books, maybe you saw it in the past, very popular.

Instead (equivalently!) we defined a second way of saying what the expected value of a random variable is:

$$E(X) = \sum_{\omega \in S} X(\omega)P(\{\omega\})$$

This second way of thinking about  $E(X)$  lends itself really nicely to taking expected value of the sum of several random variables. In particular, it lets us see that expectations are linear. Informally, this means that the expected value of a sum of random variables is equal to the sum of their expected values. It also means that we can pull constants (only constants!) outside of an expected value.

Let's start by thinking about a sequence of random variables  $X_1, X_2, \dots, X_n$ . Let's also consider  $a_1, a_2, \dots, a_n$  that are constants. Suppose we want to compute the expected value of  $a_1X_1 + a_2X_2 + \dots + a_nX_n$ .

$$\begin{aligned} E(a_1X_1 + a_2X_2 + \dots + a_nX_n) &= \sum_{\omega} (a_1X_1(\omega) + a_2X_2(\omega) + \dots + a_nX_n(\omega))P(\{\omega\}) \\ &= \sum_{\omega} a_1X_1(\omega)P(\{\omega\}) + \dots + \sum_{\omega} a_nX_n(\omega)P(\{\omega\}) \\ &= a_1 \sum_{\omega} X_1(\omega)P(\{\omega\}) + \dots + a_n \sum_{\omega} X_n(\omega)P(\{\omega\}) \\ &= a_1E(X_1) + a_2E(X_2) + \dots + a_nE(X_n) \end{aligned}$$

If we just use  $n = 2$ , this gives us

$$E(a_1X_1 + a_2X_2) = a_1E(X_1) + a_2E(X_2)$$

In particular, if we set  $X_2 = 1$  all the time, then  $E(X_2) = 1$ . So we get in that special case

$$E(a_1X_1 + a_2) = a_1E(X_1) + a_2$$

Looks even nicer if we just use  $a$  instead of  $a_1$  and if we use  $b$  instead of writing  $a_2$ . Let's write  $X$  instead of  $X_1$ . So we have the useful formula

$$E(aX + b) = aE(X) + b$$

One more nice fact: If  $A$  is an event and  $X$  is an indicator for that event, i.e., if  $X = 1$  when  $A$  occurs and  $X = 0$  otherwise, then  $E(X) = 1P(X = 1) + 0P(X = 0) = P(X = 1) = P(A)$ . So in summary, if  $X$  indicates whether or not  $A$  occurred, then

$$E(X) = P(A).$$

We will use this a lot.