Example: Take a 6-sided die and roll it as many times as necessary, to see a certain value appear for the first time, e.g., roll the die until the first "3" appears. Let $X$ denote the number of rolls that are needed. Find $E(X)$.

Idea: Define $X_j = 1$ if $j$ or more rolls are needed, to see the first "3"  
0 otherwise.

In other words, $X_j$ is an indicator random variable for the event that $j$ or more rolls are needed.

Claim $X = X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7 + \cdots$

Why? E.g., say first "3" occurs on roll 5.

- $X_1 = 1$ since 1 roll needed
- $X_2 = 1$ since 2 rolls needed
- $X_3 = 1$ since 3 rolls needed
- $X_4 = 1$ since 4 rolls needed
- $X_5 = 1$ since 5 rolls needed
- $X_6 = 0$ since 6 rolls NOT needed
- $X_7 = 0$ since 7 rolls NOT needed

\[ X = 1 + 1 + 1 + 1 + 0 + 0 + 0 + 0 + \cdots = 5 \text{ in this example.} \]

In general in this problem, $X = X_1 + X_2 + X_3 + \cdots$

$E(X) = E(X_1 + X_2 + X_3 + \cdots)$

$= E(X_1) + E(X_2) + E(X_3) + \cdots$

$= 1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3 + \cdots$

$E(X_j) = P(A_j)$ where $A_j$ indicates if $j$ or more rolls needed. happens if and only if $j-1$ rolls not successful.

$E(X_j) = \left(\frac{5}{6}\right)^{j-1}$

Notice: no calculus used  
no quotient rule  
no derivatives, etc.!

Simple?

$E(X) = \frac{1}{\frac{5}{6}} = \frac{1}{\left(1 - \frac{5}{6}\right)}$

\[ = \frac{6}{1 - \left(\frac{5}{6}\right)} = \left(1 - \left(\frac{5}{6}\right)\right)^{-1} \]

\[ = \boxed{6} \text{ just as before!} \]