

Using derivatives to find expected value and variance of Geometric random variables. Say X is a Geometric(p) random variable.

$$E(X) = \sum_{j=1}^{\infty} j P(X=j) = \sum_{j=1}^{\infty} j q^{j-1} p = p \sum_{j=1}^{\infty} \frac{d}{dq} q^j = p \frac{d}{dq} \left(\sum_{j=1}^{\infty} q^j \right) = p \frac{d}{dq} \frac{q}{1-q} = p \cdot \frac{1}{p^2} = \left(\frac{1}{p} \right)$$

note $j q^{j-1} = \frac{d}{dq} q^j$ quotient rule: $\frac{d}{dq} \frac{q}{1-q} = \frac{(1-q)(1) - (q)(-1)}{(1-q)^2} = \frac{1}{(1-q)^2} = \frac{1}{p^2}$

$$Var(X) = E(X^2) - (E(X))^2 = E((X(X-1)) + E(X) - (E(X))^2$$

\uparrow $\leftarrow \frac{1}{p}$ $\leftarrow \frac{1}{p}$ $\leftarrow \left(\frac{1}{p} \right)^2$

$$E((X(X-1)) + X)$$

$$E(X(X-1)) = \sum_{j=1}^{\infty} (j)(j-1) \frac{P(X=j)}{q^{j-1} p} = pq \sum_{j=1}^{\infty} (j)(j-1) q^{j-2} = pq \sum_{j=1}^{\infty} \frac{d^2}{dq^2} q^j = pq \frac{d^2}{dq^2} \frac{q}{1-q} = pq \frac{d}{dq} \frac{1}{(1-q)^2} = pq (-2)(1-q)^{-3} \cdot (-1) = \frac{2pq}{(1-q)^3} = \frac{2pq}{p^3} = \frac{2q}{p^2}$$

$$Var(X) = E((X(X-1)) + E(X) - (E(X))^2 = \frac{2q}{p^2} + \frac{1}{p} - \left(\frac{1}{p} \right)^2 = \frac{2(1-p) + p - 1}{p^2} = \frac{1-p}{p^2} = \frac{q}{p^2}$$