Sums of independent Poisson random variables are Poisson random variables. Let X and Y be independent Poisson random variables with parameters λ_1 and λ_2, respectively.

Define $\lambda = \lambda_1 + \lambda_2$ and $Z = X + Y$. Claim that Z is a Poisson random variable with parameter λ. Why?

$$\begin{aligned}
p_Z(z) &= P(Z = z) \\
&= \sum_{j=0}^{z} P(X = j \ \& \ Y = z - j) && \text{so } X + Y = z \\
&= \sum_{j=0}^{z} P(X = j)P(Y = z - j) && \text{since } X \text{ and } Y \text{ are independent} \\
&= \sum_{j=0}^{z} \frac{e^{-\lambda_1}\lambda_1^j}{j!} \frac{e^{-\lambda_2}\lambda_2^{z-j}}{(z-j)!} \\
&= \sum_{j=0}^{z} \frac{1}{j!(z-j)!} e^{-\lambda_1}\lambda_1^j e^{-\lambda_2}\lambda_2^{z-j} \\
&= \sum_{j=0}^{z} \frac{z!}{j!(z-j)!} \frac{e^{-\lambda_1}\lambda_1^j e^{-\lambda_2}\lambda_2^{z-j}}{z!} && \text{multiply and divide by } z! \\
&= \sum_{j=0}^{z} \binom{z}{j} \frac{e^{-\lambda_1}\lambda_1^j e^{-\lambda_2}\lambda_2^{z-j}}{z!} && \text{using the form of binominal coefficients} \\
&= \frac{e^{-\lambda}}{z!} \sum_{j=0}^{z} \binom{z}{j} \lambda_1^j \lambda_2^{z-j} && \text{factoring out } z! \text{ and } e^{-\lambda_1}e^{-\lambda_2} = e^{-\lambda_1-\lambda_2} = e^{-\lambda} \\
&= \frac{e^{-\lambda}}{z!}(\lambda_1 + \lambda_2)^z && \text{using binomial expansion (in reverse)} \\
&= \frac{e^{-\lambda}\lambda^z}{z!}
\end{aligned}$$

So altogether we showed that $p_Z(z) = \frac{e^{-\lambda}\lambda^z}{z!}$. So $Z = X + Y$ is Poisson, and we just sum the parameters.

What about a sum of more than two independent Poisson random variables? Say X_1, X_2, X_3 are independent Poissons? Then $(X_1 + X_2)$ is Poisson, and then we can add on X_3 and still have a Poisson random variable. So $X_1 + X_2 + X_3$ is a Poisson random variable. Works in general. Once we know it for two, we can keep adding more and more of them. So if $X_1, X_2, \ldots, X_n$ are independent Poisson random variables with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $X_1 + X_2 + \ldots + X_n$ is a Poisson random variable too, with parameter $\lambda_1 + \lambda_2 + \ldots + \lambda_n$.