

Sums of independent Poisson random variables are Poisson random variables. Let X and Y be independent Poisson random variables with parameters λ_1 and λ_2 , respectively.

Define $\lambda = \lambda_1 + \lambda_2$ and $Z = X + Y$. Claim that Z is a Poisson random variable with parameter λ . Why?

$$\begin{aligned}
 p_Z(z) &= P(Z = z) \\
 &= \sum_{j=0}^z P(X = j \ \& \ Y = z - j) \quad \text{so } X + Y = z \\
 &= \sum_{j=0}^z P(X = j)P(Y = z - j) \quad \text{since } X \text{ and } Y \text{ are independent} \\
 &= \sum_{j=0}^z \frac{e^{-\lambda_1} \lambda_1^j}{j!} \frac{e^{-\lambda_2} \lambda_2^{z-j}}{(z-j)!} \\
 &= \sum_{j=0}^z \frac{1}{j!(z-j)!} e^{-\lambda_1} \lambda_1^j e^{-\lambda_2} \lambda_2^{z-j} \\
 &= \sum_{j=0}^z \frac{z!}{j!(z-j)!} \frac{e^{-\lambda_1} \lambda_1^j e^{-\lambda_2} \lambda_2^{z-j}}{z!} \quad \text{multiply and divide by } z! \\
 &= \sum_{j=0}^z \binom{z}{j} \frac{e^{-\lambda_1} \lambda_1^j e^{-\lambda_2} \lambda_2^{z-j}}{z!} \quad \text{using the form of binominal coefficients} \\
 &= \frac{e^{-\lambda}}{z!} \sum_{j=0}^z \binom{z}{j} \lambda_1^j \lambda_2^{z-j} \quad \text{factoring out } z! \text{ and } e^{-\lambda_1} e^{-\lambda_2} = e^{-\lambda_1 - \lambda_2} = e^{-\lambda} \\
 &= \frac{e^{-\lambda}}{z!} (\lambda_1 + \lambda_2)^z \quad \text{using binomial expansion (in reverse)} \\
 &= \frac{e^{-\lambda} \lambda^z}{z!}
 \end{aligned}$$

So altogether we showed that $p_Z(z) = \frac{e^{-\lambda} \lambda^z}{z!}$. So $Z = X + Y$ is Poisson, and we just sum the parameters.

What about a sum of more than two independent Poisson random variables? Say X_1, X_2, X_3 are independent Poissons? Then $(X_1 + X_2)$ is Poisson, and then we can add on X_3 and still have a Poisson random variable. So $X_1 + X_2 + X_3$ is a Poisson random variable. Works in general. Once we know it for two, we can keep adding more and more of them. So if X_1, X_2, \dots, X_n are independent Poisson random variables with parameters $\lambda_1, \lambda_2, \dots, \lambda_n$, then $X_1 + X_2 + \dots + X_n$ is a Poisson random variable too, with parameter $\lambda_1 + \lambda_2 + \dots + \lambda_n$.