Variance of a hypergeometric random variable

In our recent example, $N = 20$, $M = 5$, $N - M = 15$, $n = 3$

$E(X^2) = \frac{3}{4} + \frac{6}{19}$ $\quad E(X) = \frac{3}{4}$ $\quad Var(X) = \frac{3}{4} + \frac{6}{19} - \left(\frac{3}{4}\right)^2 = \frac{153}{304}$

$= 0.5033\ldots$

General argument goes similarly

Write $X = X_1 + X_2 + \cdots + X_n$ where $X_j = 1$ if jth item is desirable, $X_j = 0$ otherwise

$E(X_i X_i) = E(X_i)$ (again $1 \cdot 1 = 1$, $0 \cdot 0 = 0$)

$= P(X_i = 1)$

$= \frac{M}{N}$

$E(X_i X_j) = 1 P(X_i X_j = 1) + \cancel{0 P(X_i X_j = 0)}$

$= P(X_i = 1 \text{ and } X_j = 1)$

$= P(X_i = 1) P(X_j = 1 \mid X_i = 1)$

$= \left(\frac{M}{N}\right)\left(\frac{M-1}{N-1}\right)$

$E(X) = n \frac{M}{N}$ $\quad E(X^2) = n E(X_1 X_1) + \underbrace{(n)(n-1)}_{n^2 - n = (n)(n-1)} E(X_1 X_2)$

diagonal entries all the same

off diagonal entries, all the same

$$E(X^2) = n \frac{M}{N} + (n)(n-1)\left(\frac{M}{N}\right)\left(\frac{M-1}{N-1}\right)$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$= n \frac{M}{N} + (n)(n-1)\left(\frac{M}{N}\right)\left(\frac{M-1}{N-1}\right) - \left(n \frac{M}{N}\right)^2$$

Simplify

$$= n \frac{M}{N}\left(1 - \frac{M}{N}\right)\left(\frac{N-n}{N-1}\right) = \text{Variance of a hypergeometric}$$

$n \frac{M}{N}$ = expected value of a hypergeometric