Example

Suppose \( X, Y \) have joint density \( f_{X,Y}(x,y) = \frac{2}{9} \) for

\[
x > 0, \quad y > 0, \quad x + \frac{y}{2} \leq 3
\]

\( f_{X,Y}(x,y) = 0 \) otherwise

\[
\int_{0}^{3} \int_{0}^{3-x} \frac{2}{9} \, dy \, dx = \frac{2}{9} \cdot \left( \text{area of } \square \right) = \left( \frac{2}{9} \right) \left( \frac{9}{2} \right) = 1.
\]

\[
\Pr(X \leq 2) = \int_{2}^{3} \int_{0}^{3-x} \frac{2}{9} \, dy \, dx = \frac{2}{9} \cdot \left( \text{area of } \square \right) = 2 \cdot \frac{8}{9} = \frac{16}{9}
\]

\[
\text{Q: Are } X, Y \text{ indep? No, because the joint density is not defined to be nonzero in a rectangular grid or series of rectangular grids.}
\]

What is the density of \( X \)? (Symmetric: Know the density of \( Y \) is similar.)

\[
f_X(x) = \int f_{X,Y}(x,y) \, dy
\]

For \( 0 \leq x < 3 \),

\[
f_X(x) = \int_{0}^{3-x} \frac{2}{9} \, dy = \frac{2}{9} \left( 3-x \right) = \frac{2}{9} \left( 3-x \right).
\]

Check that this is a valid density:

\[
f_X(x) = \begin{cases} \frac{2}{9} (3-x) & \text{for } 0 \leq x < 3 \\ 0 & \text{otherwise} \end{cases}
\]

Always nonnegative. Does it integrate to 1?

\[
\int_{0}^{3} \frac{2}{9} (3-x) \, dx = \frac{2}{9} \left[ (3x - \frac{x^2}{2}) \right]_{x=0}^{3} = \frac{2}{9} \left( 9 - \frac{9}{2} \right) = \frac{2}{9} \left( \frac{9}{2} \right) = 1.
\]

By symmetry, we get the density of \( Y \) too:

\[
f_Y(y) = \begin{cases} \frac{2}{9} (3-y) & \text{for } 0 \leq y < 3 \\ 0 & \text{otherwise} \end{cases}
\]

One last comment: \( X, Y \) not indep because \( f_{XY}(xy) \) not defined in rectangles, but also not independent since \( f_{X,Y}(x,y) \neq f_X(x) f_Y(y) \) i.e. \( \frac{2}{9} \neq \frac{2}{9} (3-x) \frac{2}{9} (3-y) \).