

Hat problem: Suppose 10 people attend a party, each checks their hat at the door, they pick up a random hat when they leave, all choices equally likely.

Let  $X$  indicate whether Alice get her back, i.e.  $X=1$  if she does,  $X=0$  otherwise.

Let  $Y$  indicate whether Bob gets his hat back:  $Y=1$  if so,  $Y=0$  otherwise.

Find  $\text{Var}(X+Y)$

$$\begin{aligned} \text{Var}(X+Y) &= \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y) \\ &= \binom{1}{10} \binom{9}{10} + \binom{1}{10} \binom{9}{10} + 2 \left( \binom{1}{10} \binom{1}{9} - \binom{1}{10} \binom{1}{10} \right) \\ &= \frac{41}{225} \end{aligned}$$

$$\text{Cov}(X, Y) = \binom{1}{10} \binom{1}{9} - \binom{1}{10} \binom{1}{10}$$

$$E(XY) - E(X)E(Y)$$

$$\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow$$

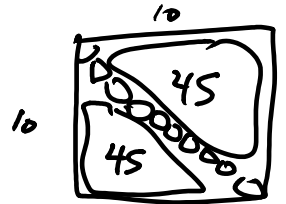
$$P(XY=1) - \underbrace{P(X=1)}_{\frac{1}{10}} \underbrace{P(Y=1)}_{\frac{1}{10}}$$

$$= P(X=1 \text{ and } Y=1) = P(X=1)P(Y=1|X=1) = \left(\frac{1}{10}\right) \left(\frac{1}{9}\right)$$

Extend: Let  $X_j = 1$  if  $j$ th person gets her/his hat back,  $X_j = 0$  otherwise  
 So  $X_1 + \dots + X_{10}$  is the total # of people who get their own hat back.

$$E(X_1 + \dots + X_{10}) = E(X_1) + \dots + E(X_{10}) = \frac{1}{10} + \dots + \frac{1}{10} = 1.$$

$$\begin{aligned} \text{Var}(X_1 + \dots + X_{10}) &= \sum_{i=1}^{10} \text{Var}(X_i) + 2 \sum_{i < j} \text{Cov}(X_i, X_j) \\ &= (10) \left(\frac{1}{10}\right) \left(\frac{9}{10}\right) + (90) \left( \binom{1}{10} \binom{1}{9} - \binom{1}{10} \binom{1}{10} \right) \\ &= 1 \end{aligned}$$



all 100 pairs except those 10 on the diagonal handled here