Hat problem: Suppose 10 people attend a party, each checks their hat at the door, they pick up a random hat when they leave, all choices equally likely.

Let $X$ indicate whether Alice gets her back, i.e. $X=1$ if she does, $X=0$ otherwise.

Let $Y$ indicate whether Bob gets his hat back: $Y=1$ if so, $Y=0$ otherwise.

Find $\text{Var}(X + Y)$

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2 \text{Cov}(X, Y)$$

$$= \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^2 + 2\left(\frac{1}{10}\frac{(1/10)^2}{10} - \frac{1}{10}\frac{1}{10}\right)$$

$$= \frac{4}{225}$$

$E(10) = \frac{1}{10}$

$P(X=1) = P(X=1 \text{ and } Y=1) P(Y=1) = P(X=1 \text{ and } Y=1) = \frac{1}{10} \frac{1}{10}$

Extend: Let $X_j = 1$ if jth person gets her/his hat back, $X_j = 0$ otherwise.

So $X_1 + \cdots + X_{10}$ is the total # of people who get their own hat back.

$E(X_1 + \cdots + X_{10}) = E(X_1) + \cdots + E(X_{10}) = \frac{1}{10} + \cdots + \frac{1}{10} = 1.$

$\text{Var}(X_1 + \cdots + X_{10}) = \sum_{i} \text{Var}(X_i) + 2 \sum_{i<j} \text{Cov}(X_i, X_j)$

$$= \left(\frac{1}{10}\right)^2 + \frac{1}{10} \left(\frac{1}{10}\right)^2 + \frac{1}{10} \left(\frac{1}{10}\right)^2 + \frac{1}{10} \left(\frac{1}{10}\right)^2$$

$$= 1$$

all 100 pairs except those 10 in the diagonal handled here