

Example with continuous random variables

Say X, Y have joint probability density $f_{X,Y}(x,y) = 40e^{-3x-5y}$ for $0 < x < y$
 $= 0$ otherwise.

Find $\text{Var}(X+Y)$.

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X,Y)$$

$$E(X^2) - (E(X))^2$$

$$\int_0^{\infty} \int_x^{\infty} (x^2)(40e^{-3x-5y}) dy dx = \frac{1}{32} \quad \left(\int_0^{\infty} \int_x^{\infty} (x)(40e^{-3x-5y}) dy dx \right)^2 = \left(\frac{1}{8}\right)^2$$

$$\text{Var}(X) = \frac{1}{32} - \left(\frac{1}{8}\right)^2.$$

$$\text{Var}(Y) = E(Y^2) - (E(Y))^2$$

$$\int_0^{\infty} \int_x^{\infty} (y^2)(40e^{-3x-5y}) dy dx = \frac{129}{800} \quad \left(\int_0^{\infty} \int_x^{\infty} (y)(40e^{-3x-5y}) dy dx \right)^2 = \left(\frac{13}{40}\right)^2$$

$$\text{Var}(Y) = \frac{129}{800} - \left(\frac{13}{40}\right)^2$$

$$\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$\int_0^{\infty} \int_x^{\infty} (xy)(40e^{-3x-5y}) dy dx = \frac{9}{160}$$

$$\text{Cov}(X,Y) = \frac{9}{160} - \left(\frac{1}{8}\right)\left(\frac{13}{40}\right)$$

Put it all together:

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X,Y) = \frac{41}{400}.$$