General results about the density of a specific order statistic

Consider $U_1, U_2, \ldots, U_n$ that are independent, continuous Uniform random variables on $[a, b]$, and suppose we want the density of $U_{(j)}$ i.e. of the jth smallest of the U's.

$$f_{U_{(j)}}(u) = \binom{n}{j-1,\,1,\,n-j}\left(\frac{1}{b-a}\right)\left(\frac{u-a}{b-a}\right)^{j-1}\left(1-\frac{u-a}{b-a}\right)^{n-j} \quad \text{for } a<u<b$$

$$\frac{n!}{(j-1)!\,1!\,(n-j)!}$$

In particular if $[a,b] = [0,c]$

$$f_{U_{(j)}}(u) = \binom{n}{j-1,\,1,\,n-j}\left(\frac{1}{c}\right)\left(\frac{u}{c}\right)^{j-1}\left(1-\frac{u}{c}\right)^{n-j} \quad \text{for } 0<u<c$$

Now consider $X_1, X_2, \ldots, X_n$ which are independent Exponential random variables, each with parameter λ i.e. $E(X_j) = \frac{1}{\lambda}$ for each j.

Find the density of $X_{(j)}$ i.e. of the jth order statistic:

$$f_{X_{(j)}}(x) = \binom{n}{j-1,\,1,\,n-j}(\lambda e^{-\lambda x})(1-e^{-\lambda x})^{j-1}(e^{-\lambda x})^{n-j} \quad \text{for } x>0$$

Notice: If $j \neq 1$, then $X_{(j)}$ is not exponential

If $j=1$ i.e. if we look at the min,

$$f_{X_{(1)}}(x) = \binom{n}{0,\,1,\,n-1}(\lambda e^{-\lambda x})(1-e^{-\lambda x})^{0}(e^{-\lambda x})^{n-1} = n\lambda e^{-n\lambda x} \quad \text{for } x>0$$

$$\frac{n!}{(n-1)!} = n$$

So $X_{(1)}$ i.e. the minimum of $X_1, \ldots, X_n$ is an exponential random variable with parameter $n\lambda$ i.e. $E(X_{(1)}) = \frac{1}{n\lambda}$.