

Moment Generating Function of a Binomial random variable X with parameters n and p .

$$\begin{aligned}
 M_X(t) &= E(e^{tX}) = \sum_x e^{tx} p_X(x) = \sum_{x=0}^n e^{tx} \binom{n}{x} p^x (1-p)^{n-x} \\
 &= \sum_{x=0}^n \binom{n}{x} (e^t p)^x (1-p)^{n-x} \\
 &= (e^t p + 1-p)^n
 \end{aligned}$$

Use Binomial THM:

$$\sum_{j=0}^n \binom{n}{j} a^j b^{n-j} = (a+b)^n$$

use x instead of j
 $e^t p$ instead of a
 $1-p$ instead of b

$$\begin{aligned}
 E(X) &= \frac{\partial}{\partial t} M_X(t) \Big|_{t=0} = M_X'(0) \\
 &= np \text{ which is } E(X) \text{ as we know.}
 \end{aligned}$$

$$\begin{aligned}
 E(X^2) &= \frac{\partial^2}{\partial t^2} M_X(t) \Big|_{t=0} = M_X''(0) \\
 &= (n)(n-1)p^2 + np
 \end{aligned}$$

$$\begin{aligned}
 \text{Var}(X) &= E(X^2) - (E(X))^2 = (n)(n-1)p^2 + np - (np)^2 \\
 &= np(1-p) \text{ as we know } \checkmark
 \end{aligned}$$