

STAT/MA 41600
Mid-September Review: September 24, 2014
Solutions by Mark Daniel Ward

1. Write E_i for the event that Urn i is selected. Write B_1 for the event that the first ball drawn is black, and write B_2 for the event that the second ball drawn is black. Then the desired probability is

$$P(B_2 | B_1) = \frac{P(B_1 \cap B_2)}{P(B_1)}$$

which can be rewritten as

$$\frac{P(B_1 \cap B_2 | E_1)P(E_1) + P(B_1 \cap B_2 | E_2)P(E_2) + P(B_1 \cap B_2 | E_3)P(E_3) + P(B_1 \cap B_2 | E_4)P(E_4)}{P(B_1 | E_1)P(E_1) + P(B_1 | E_2)P(E_2) + P(B_1 | E_3)P(E_3) + P(B_1 | E_4)P(E_4)}$$

which simplifies to

$$\frac{(0)(1/4) + (0)(1/4) + (1)(1/4) + (0)(1/4)}{(1/2)(1/4) + (1/2)(1/4) + (1)(1/4) + (0)(1/4)} = \frac{1}{2}$$

So the correct answer is **E**, namely, $1/2$.

A simple way to look at the problem is this: There are 4 black balls; each of these four is equally likely to be picked on the first draw. Exactly two of these four balls are from Urn 3, so choosing exactly two of these four balls will make the second ball also be black. So the desired conditional probability is $2/4 = 1/2$.

2. Any of the 10 balls is equally likely to be drawn on the fourth time. So the probability that the fourth ball is black is $4/10 = 2/5$. So the correct answer is **D**, namely, $2/5$.

3. Write E for the event that the student knew the answer to a question, and write F for the event that he or she answered it correctly. So the desired probability is

$$P(E | F) = \frac{P(E \cap F)}{P(F)} = \frac{P(F | E)P(E)}{P(F | E)P(E) + P(F | E^c)P(E^c)} = \frac{(1)(.70)}{(1)(.70) + (1/5)(.30)} \approx .92$$

So the correct answer is **E**, namely, .92.

4. Write E_i for the event that Urn i was selected, and write F for the event that a black ball was drawn. Then the desired probability is

$$\begin{aligned} P(E_1 | F) &= \frac{P(E_1 \cap F)}{P(F)} = \frac{P(F | E_1)P(E_1)}{P(F | E_1)P(E_1) + P(F | E_2)P(E_2) + P(F | E_3)P(E_3)} \\ &= \frac{(2/4)(1/3)}{(2/4)(1/3) + (1/3)(1/3) + (3/4)(1/3)} = 6/19 \end{aligned}$$

So the correct answer is **B**, namely, $6/19$.

5. With replacement, there are two possible methods of calculating the desired probability. First, we just directly write the sum of the probabilities that B wins on his first, second, third, etc., try:

$$\begin{aligned} P(B \text{ wins}) &= \binom{5}{7} \binom{2}{7} + \binom{5}{7} \binom{5}{7} \binom{5}{7} \binom{2}{7} + \binom{5}{7} \binom{5}{7} \binom{5}{7} \binom{5}{7} \binom{5}{7} \binom{2}{7} + \cdots \\ &= \sum_{n=0}^{\infty} \left(\binom{5}{7} \binom{5}{7} \right)^n \binom{5}{7} \binom{2}{7} \\ &= \binom{5}{7} \binom{2}{7} \frac{1}{1 - \left(\frac{5}{7} \right) \left(\frac{5}{7} \right)} \\ &= 5/12 \end{aligned}$$

Another possible method of solution when the balls are replaced is the following: The probability that a round (i.e., A then B) will complete the game is $\binom{2}{7} \binom{2}{7} + \binom{2}{7} \binom{5}{7} + \binom{5}{7} \binom{2}{7}$, and of these three possibilities, exactly one corresponds to the probability that B is the one of complete the game, namely $\binom{5}{7} \binom{2}{7}$. So the probability that B wins is

$$\frac{\binom{5}{7} \binom{2}{7}}{\binom{2}{7} \binom{2}{7} + \binom{2}{7} \binom{5}{7} + \binom{5}{7} \binom{2}{7}} = 5/12$$

When the balls are withdrawn without replacement, we just calculate directly the probability that B wins, namely:

$$\binom{5}{7} \binom{2}{6} + \binom{5}{7} \binom{4}{6} \binom{3}{5} \binom{2}{4} + \binom{5}{7} \binom{4}{6} \binom{3}{5} \binom{2}{4} \binom{1}{3} \binom{2}{2} = 3/7$$

So the correct answer is **B**, namely, with replacement, $P(B \text{ wins}) = 5/12$; without replacement, $P(B \text{ wins}) = 3/7$.

6. Write H for the event that the coin shows heads. Write E_1 for the event that die A is used, and write E_2 for the event that die B is used. Write F for the event that the first n throws of the die are red. Then the desired probability is

$$\begin{aligned} P(E_1 | F) &= \frac{P(E_1 \cap F)}{P(F)} \\ &= \frac{P(F | E_1)P(E_1)}{P(F | E_1)P(E_1) + P(F | E_2)P(E_2)} \\ &= \frac{(4/6)^n(1/2)}{(4/6)^n(1/2) + (2/6)^n(1/2)} \\ &= \frac{(2/3)^n(1/2)}{(2/3)^n(1/2) + (1/3)^n(1/2)} \end{aligned}$$

We simplify by multiplying the numerator and denominator by 2 and also by 3^n . So the desired probability is $\frac{2^n}{2^n+1}$.

So the correct answer is **D**, namely, $\frac{2^n}{2^n+1}$.

7. Let E_n denote the event that at least one of the first n rolls show “5”; thus E_n^c is the event that none of the first n rolls show “5”, and $P(E_n^c) = (5/6)^n$. We compute

$$\frac{9}{10} \leq P(E_n) = 1 - P(E_n^c) = 1 - (5/6)^n$$

Thus we need $(5/6)^n \leq 1/10$, i.e., $n \ln(5/6) \leq \ln(1/10)$. Dividing by $\ln(5/6)$ (which is a negative number) reverses the inequality, and we obtain $n \geq \ln(1/10)/\ln(5/6) \approx 12.63$.

So the correct answer is **D**, namely, 13.

8. Considering the various outcomes, we notice that

$$P(X = i) = \begin{cases} 1/36 & i = 1 \\ 3/36 & i = 2 \\ 5/36 & i = 3 \\ 7/36 & i = 4 \\ 9/36 & i = 5 \\ 11/36 & i = 6 \end{cases}$$

Thus

$$E[X] = (1)(1/36) + (2)(3/36) + (3)(5/36) + (4)(7/36) + (5)(9/36) + (6)(11/36) = 161/36 \approx 4.47$$

So the correct answer is **D**, namely, 4.47.