STAT/MA 41600
Practice Problems: August 29, 2014
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1. Choosing a page at random.

Let A_1 be the event that the ones digit is a "5"; let A_2 be the event that the tens digit is a "5"; let A_3 be the event that the hundreds digit is a "5". Then we want to find

$$P(A_1 \cup A_2 \cup A_3).$$

By inclusion-exclusion, we have

$$\begin{aligned}
P(A_1 \cup A_2 \cup A_3) &= P(A_1) + P(A_2) + P(A_3) \\
&\qquad - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3) \\
&= \frac{1}{10} + \frac{1}{10} + \frac{1}{10} - \frac{1}{100} - \frac{1}{100} - \frac{1}{100} + \frac{1}{1000} \\
&= 271/1000.
\end{aligned}$$

2. Gloves. (a.) For A_j to occur, he must miss the white glove exactly $j-1$ times, and then finally get the white glove. So $P(A_j) = \left(\frac{4}{5}\right)^{j-1}\left(\frac{1}{5}\right)$.

(b.) *Method #1:*

$$\begin{aligned}
P(B_1) &= \frac{1}{5} \\
P(B_2) &= \frac{(4)(1)}{(5)(4)} = \frac{1}{5} \\
P(B_3) &= \frac{(4)(3)(1)}{(5)(4)(3)} = \frac{1}{5} \\
P(B_4) &= \frac{(4)(3)(2)(1)}{(5)(4)(3)(2)} = \frac{1}{5} \\
P(B_5) &= \frac{(4)(3)(2)(1)(1)}{(5)(4)(3)(2)(1)} = \frac{1}{5}
\end{aligned}$$

Method #2: We could just notice that, if he continues to pull all the gloves out of the drawer until it is empty, the white glove is equally likely to appear in any of the 5 positions. Thus $P(B_k) = 1/5$ for each k.

3. Seating arrangements. *Method #1:* Alice sits somewhere. Someone, say a "mystery" person, sits on Alice's right-hand side. If the mystery person's spouse sits beside him/her, then both couples are happy; this happens with probability 1/3, so $P(A_2) = 1/3$. If the mystery person's spouse sits one seat away from him/her, then neither couple is happy; this happens with probability 1/3, so $P(A_0) = 1/3$. If the mystery person sits two seats away from him/her (i.e., all the way around the table, on Alice's other side), then only the other couple is happy; this happens with probability 1/3, so $P(A_1) = 1/3$.

Method #2: Alice sits somewhere. For the couples to sit together, either Bob and Catherine (collectively, as a pair) sit on Alice's left or right. In either case, Bob and Catherine have two ways of sitting among themselves (i.e., within their own pair). Within the two remaining seats, Doug and Edna have two ways of sitting among themselves (i.e., within their own pair). So there are $2 \times 2 \times 2 = 8$ possible ways that the couples can sit happily. So $P(A_2) = 8/24 = 1/3$. For neither couple to sit together, someone sits on Alice's left (4 choices), and then his/her spouse must be 2 chairs away (1 choice), and then there are two ways that the remaining couple can be seated in the two remaining chairs (2 choices). Thus $P(A_0) = 8/24 = 1/3$. For exactly one couple to sit together, someone sits on Alice's left (4 choices), and then his/her spouse must be on Alice's right (1 choice), and then there are two ways that the remaining couple can be seated together in the two remaining chairs (2 choices). Thus $P(A_1) = 8/24 = 1/3$.

4. Abstract art.

EVENT:	Probability of the event:
$\{(P)\}$	1/3, since Purple is first 1/3 of the time.
$\{(G, P), (Y, P)\}$	1/3, since Purple is second 1/3 of the time
$\{(G, P), (G, Y, P)\}$	1/3, since Green is first 1/3 of the time
$\{(Y, G, P), (G, Y, P)\}$	1/3, since Purple is last 1/3 of the time

$\{(P), (Y, P)\}$ $1/3 + 1/6 = 1/2$. The first probability is 1/3 since Purple is first 1/3 of the time. The second probability is 1/6 since there is only 1 out of $(3)(2) = 6$ ways to pick Yellow then Purple. The events in the two previous sentences are disjoint, so we can just add their probabilities.

5. Maximum of three dice. Event B_k occurs if and only if all three rolls are between 1 and k. So B_k occurs in $k \times k \times k = k^3$ ways out of the $6^3 = 216$ ways altogether. Thus $P(B_k) = \frac{k^3}{216} = (k/6)^3$.