STAT/MA 41600
Practice Problems: September 5, 2014
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1. Choosing a page at random. (a.) We have $P(A \mid B) = \frac{P(A\cap B)}{P(B)}$. Since $A \subset B$, then $P(A \cap B) = P(A) = 1/1000$. Also $P(B) = 271/1000$, as discovered on problem set 2. Thus $P(A \mid B) = \frac{1/1000}{271/1000} = 1/271$.

(b.) We have $P(A \mid C) = \frac{P(A\cap C)}{P(C)}$. Since $A \subset C$, then $P(A \cap C) = P(A) = 1/1000$.

We compute $P(C)$ using inclusion-exclusion, as in problem set 2. Let C_1 be the event that the ones and tens digits are "5"s; let C_2 be the event that the tens and hundreds digits are "5"s; let C_3 be the event that the one and hundreds digits are "5"s. Then we want $P(C) = P(C_1 \cup C_2 \cup C_3)$. By inclusion-exclusion, we have

$$\begin{aligned} P(C_1 \cup C_2 \cup C_3) &= P(C_1) + P(C_2) + P(C_3) \\ &\qquad - P(C_1 \cap C_2) - P(C_1 \cap C_3) - P(C_1 \cap C_3) + P(C_1 \cap C_2 \cap C_3) \\ &= \frac{1}{100} + \frac{1}{100} + \frac{1}{100} - \frac{1}{1000} - \frac{1}{1000} - \frac{1}{1000} + \frac{1}{1000} \\ &= 28/1000. \end{aligned}$$

Thus $P(A \mid C) = \frac{1/1000}{28/1000} = 1/28$.

2. Gloves. (a.) As before, $P(A \mid B) = \frac{P(A\cap B)}{P(B)}$. Also $A \subset B$, so $P(A \cap B) = P(A)$. The probability both gloves are blue is $P(A) = (\frac{2}{5})(\frac{1}{4}) = \frac{1}{10}$. The probability neither glove is blue is $(\frac{3}{5})(\frac{2}{4}) = \frac{3}{10}$; the complementary probability is $P(B) = 1 - 3/10 = 7/10$. Thus $P(A \mid B) = \frac{1/10}{7/10} = 1/7$.

(b.) Since conditional probabilities satisfy all of the three standard axioms of probability, then $P(A^c \mid B) = 1 - P(A \mid B) = 1 - 1/7 = 6/7$.

3. Seating arrangements. Given that Bob and Catherine are sitting next to each other, there are three remaining consecutive seats in a row. If Alice sits in the middle, then Doug and Edna are separated; this happens with probability 1/3. If Alice sits on either end of the three remaining seats, then Doug and Edna are together; this happens with probability 2/3. Thus $P(A \mid B) = 2/3$.

4. Pair of dice. Exactly 6 out of the 36 equally likely outcomes have the same result on both die (i.e., are "doubles"). Thus, $P(B) = 30/36$ is the probability that the values on the dice do not agree.

Also, exactly 18 out of 36 of the outcomes are even; exactly 6 of these outcomes are "doubles", and the other 12 have even sums but are not doubles. So $P(A \cap B) = 12/36$.

So we conclude $P(A \mid B) = \frac{P(A\cap B)}{P(B)} = \frac{12/36}{30/36} = 12/30 = 2/5$.

5. Pair of dice. Exactly 10 out of the 36 equally likely outcomes have sum 9 or larger. Thus, $P(B) = 10/36$. Also $A \subset B$, so $P(A \cap B) = P(A)$. Also, exactly 3 out of the 36 equally likely outcomes have sum exactly 10, so $P(A) = 3/36$. Thus So we conclude $P(A \mid B) = \frac{P(A\cap B)}{P(B)} = \frac{3/36}{10/36} = 3/10$.