STAT/MA 41600
Practice Problems: September 8, 2014
Solutions by Mark Daniel Ward

1. Waking up at random. 1a. Writing A as the event it is a weekday, and B as the event it is before 8 AM, we have

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A^c)P(A^c)} = \frac{(.65)(5/7)}{(.65)(5/7) + (.22)(2/7)} = .881.$$

1b. Writing A as the event it is a weekday, and B as the event it is after 8 AM, we have

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A^c)P(A^c)} = \frac{(.35)(5/7)}{(.35)(5/7) + (.78)(2/7)} = .529.$$

2. Gloves. We really need to know whether the first glove is white or blue. So let B be the event that the second glove is blue. Let A be the event that the first glove is blue or white. We want $P(B \mid A) = \frac{P(A\cap B)}{P(A)}$. Of course $P(A) = 3/5$. Also $P(A\cap B) = (2/5)(1/4)+(1/5)(2/4) = 1/5$. So $P(B \mid A) = 1/3$.

3. Pair of dice. Let A be the event that the sum of the two dice is exactly 4. Let B be the event that the blue die has an odd value.

Then
$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

Notice $P(B) = 3/6 = 1/2$ since the blue die has an odd value exactly 1/2 the time. Also $P(A \cap B) = 2/36 = 1/18$ since only 2 out of the 36 equally likely outcomes are in $A \cap B$, namely, if the (blue,red) values are (1,3) or (3,1). Thus $P(A \mid B) = \frac{1/18}{1/2} = 1/9$.

4. Pair of dice. Let A be the event that the sum of the two dice is 7 or larger. Let B be the event that the blue die has a value of 4 or smaller. Then $P(A \mid B) = \frac{P(A\cap B)}{P(B)}$. Notice $P(B) = 4/6 = 2/3$.

It is easier to calculuate $P(A\cap B)$ if we break B up into four smaller events B_1, B_2, B_3, B_4, namely, the events that the blue die has value exactly 1, 2, 3, or 4, respectively. These are disjoint events, so

$$\begin{aligned}
P(A \cap B) &= P(A \cap (B_1 \cup B_2 \cup B_3 \cup B_4)) \\
&= P((A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3) \cup (A \cap B_4)) \\
&= P(A \cap B_1) + (A \cap B_2) + (A \cap B_3) + (A \cap B_4) \\
&= P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + P(A \mid B_3)P(B_3) + P(A \mid B_4)P(B_4) \\
&= (1/6)(1/6) + (2/6)(1/6) + (3/6)(1/6) + (4/6)(1/6) = 10/36 = 5/18
\end{aligned}$$

Thus $P(A \mid B) = \frac{5/18}{2/3} = 5/12$.

5. Coin flips and then dice. Let B be the event that none of the dice show the value 1, and let A_k be the event that exactly k flips are needs to get heads. Then

$$\begin{aligned}
P(B) &= P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) + P(A_4 \cap B) + \cdots \\
&= P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) + P(B \mid A_3)P(A_3) + P(B \mid A_4)P(A_4) + \cdots \\
&= (5/6)^1(1/2)^1 + (5/6)^2(1/2)^2 + (5/6)^3(1/2)^3 + (5/6)^4(1/2)^4 + \cdots \\
&= (5/12)\sum_{j=0}^{\infty}(5/12)^j \\
&= \frac{5/12}{1-5/12} \\
&= 5/7.
\end{aligned}$$