STAT/MA 41600
Practice Problems: September 15, 2014
Solutions by Mark Daniel Ward

1. Butterflies. Using the work from Problem Set 8, we have

$$p_{X,Y}(0,3) = .3424, \qquad p_{X,Y}(1,2) = .4644, \qquad p_{X,Y}(2,1) = .1741, \qquad p_{X,Y}(3,0) = .0191,$$

and otherwise $p_{X,Y}(x,y) = 0$.

2. Dependence/independence among dice rolls. Yes, X and Y are independent because for positive integers x and y, we have

$$p_{X,Y}(x,y) = \left((5/6)^{x-1}(1/6)\right)\left((5/6)^{y-1}(1/6)\right) = p_X(x)p_Y(y)$$

and otherwise $p_{X,Y}(x,y) = 0 = p_X(x)p_Y(y)$. So X and Y are independent.

3. Wastebasket basketball. Using the results from Problem Set 8, we have $p_Y(1) = 1/3 + 2/9 + 4/27 = 19/27$, and thus $p_Y(0) = 8/27$. So we have

$$p_{X|Y}(1 \mid 0) = \frac{p_{X,Y}(1,0)}{p_Y(0)} = \frac{0}{8/27} = 0$$
$$p_{X|Y}(2 \mid 0) = \frac{p_{X,Y}(2,0)}{p_Y(0)} = \frac{0}{8/27} = 0$$
$$p_{X|Y}(3 \mid 0) = \frac{p_{X,Y}(3,0)}{p_Y(0)} = \frac{0}{8/27} = 0$$
$$p_{X|Y}(4 \mid 0) = \frac{p_{X,Y}(4,0)}{p_Y(0)} = \frac{8/81}{8/27} = 1/3$$
$$p_{X|Y}(5 \mid 0) = \frac{p_{X,Y}(5,0)}{p_Y(0)} = \frac{16/243}{8/27} = 2/9$$
$$p_{X|Y}(6 \mid 0) = \frac{p_{X,Y}(6,0)}{p_Y(0)} = \frac{32/243}{8/27} = 4/9$$

and

$$p_{X|Y}(1 \mid 1) = \frac{p_{X,Y}(1,1)}{p_Y(1)} = \frac{1/3}{19/27} = 9/19$$
$$p_{X|Y}(2 \mid 1) = \frac{p_{X,Y}(2,1)}{p_Y(1)} = \frac{2/9}{19/27} = 6/19$$
$$p_{X|Y}(3 \mid 1) = \frac{p_{X,Y}(3,1)}{p_Y(1)} = \frac{4/27}{19/27} = 4/19$$
$$p_{X|Y}(4 \mid 1) = \frac{p_{X,Y}(4,1)}{p_Y(1)} = \frac{0}{19/27} = 0$$
$$p_{X|Y}(5 \mid 1) = \frac{p_{X,Y}(5,1)}{p_Y(1)} = \frac{0}{19/27} = 0$$
$$p_{X|Y}(6 \mid 1) = \frac{p_{X,Y}(6,1)}{p_Y(1)} = \frac{0}{19/27} = 0$$

4. Two 4-sided dice. In general, for $1 \leq x < y \leq 4$,

$$\begin{aligned} p_{X,Y}(x,y) &= P(\{\max = y \text{ and } \min = x\}) \\ &= P(\{\text{die values } x, y\}) + P(\{\text{die values } y, x\}) \\ &= 2/16, \end{aligned}$$

and for $1 \leq x = y \leq 4$,

$$\begin{aligned} p_{X,Y}(x,y) &= P(\{\max = y = x = \min\}) \\ &= P(\{\text{die values } x, x\}) \qquad \text{since } x \text{ and } y \text{ are the same in this case} \\ &= 1/16, \end{aligned}$$

and

$$p_{X,Y}(x,y) = 0 \qquad \text{otherwise.}$$

We have

$F_{X,Y}(x,y)$	$y=1$	$y=2$	$y=3$	$y=4$
$x=1$	1/16	3/16	5/16	7/16
$x=2$	1/16	4/16	8/16	12/16
$x=3$	1/16	4/16	9/16	15/16
$x=4$	1/16	4/16	9/16	16/16

5. Pick two cards. The random variables X and Y are dependent. As an example, we know $P(X = 2) = 11/221$; this was established back in Problem Set 7. On the other hand, given $Y = 2$, then both of the cards that you selected are 10's, so they cannot be face cards, and thus

$$P(X = 2 \mid Y = 2) = 0.$$

So we have

$$P(X = 2 \mid Y = 2) \neq P(X = 2).$$

So X and Y are dependent random variables.