1a. Variance of an Indicator. We have \( \mathbb{E}(X^2) = 1^2(p) + 0^2(1-p) = p \). So \( \text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = p - p^2 = p(1-p) \).

1b. Butterflies. Method #1: Write \( X \) as the sum of three indicator random variables, \( X_1, X_2, X_3 \) that indicate whether Alice, Bob, Charlotte (respectively) found a butterfly. Then \( X = X_1 + X_2 + X_3 \). Since \( X_1, X_2, X_3 \) are independent, then \( \text{Var}(X) = \text{Var}(X_1 + X_2 + X_3) = \text{Var}(X_1) + \text{Var}(X_2) + \text{Var}(X_3) = (.17)(.83) + (.25)(.75) + (.45)(.55) = .5761 \).

Method #2: The mass and expected value of \( X \) were given in Problem Set 10:

The mass of \( X \) is

\[
p_X(0) = 0.3424, \quad p_X(1) = 0.4644, \quad p_X(2) = 0.1741, \quad p_X(3) = 0.0191,
\]

so the expected value of \( X \) is

\[
\mathbb{E}(X) = (0)(0.3424) + (1)(0.4644) + (2)(0.1741) + (3)(0.0191) = .87.
\]

The expected value of \( X^2 \) is

\[
\mathbb{E}(X^2) = (0^2)(0.3424) + (1^2)(0.4644) + (2^2)(0.1741) + (3^2)(0.0191) = 1.33.
\]

So the variance of \( X \) is \( \text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 1.3327 - (.87)^2 = .58 \).

2. Appetizers. Method #1: The expected value of \( X \) is:

\[
\mathbb{E}(X) = (1)(.08) + (2)(.20) + (3)(.32) + (4)(.40) = 3.04,
\]

and

\[
\mathbb{E}(X^2) = 1^2(.08) + 2^2(.20) + 3^2(.32) + 4^2(.40) = 10.16,
\]

so

\[
\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 10.16 - (3.04)^2 = .9184.
\]

So

\[
\text{Var}(Y) = \text{Var}(1.07X) = (1.07)^2\text{Var}(X) = (1.07)^2(.9184) = 1.0515.
\]

Method #2: The expected value of \( Y \) is:

\[
\mathbb{E}(Y) = (1.07)(1)(.08) + (1.07)(2)(.20) + (1.07)(3)(.32) + (1.07)(4)(.40) = 3.2528,
\]

and

\[
\mathbb{E}(Y^2) = ((1.07)(1))^2(.08)+((1.07)(2))^2(.20)+((1.07)(3))^2(.32)+((1.07)(4))^2(.40) = 11.632184,
\]
so
\[ \text{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 11.632184 - (3.2528)^2 = 1.0515. \]

3. Gloves. As discussed in Problem Set 3, \( X \) has mass \( 1/5 \) on the values 1, 2, 3, 4, 5. So
\[ \mathbb{E}(X) = (1)(1/5) + (2)(1/5) + (3)(1/5) + (4)(1/5) + (5)(1/5) = 3. \]

Find \( \mathbb{E}(X^2) \).
Again using the mass of \( X \), we have
\[ \mathbb{E}(X^2) = 1^2(1/5) + 2^2(1/5) + 3^2(1/5) + 4^2(1/5) + 5^2(1/5) = 11. \]

We have
\[ \text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 11 - 3^2 = 2. \]

Again using the mass of \( X \), we have
\[ \mathbb{E}(X^3) = 1^3(1/5) + 2^3(1/5) + 3^3(1/5) + 4^3(1/5) + 5^3(1/5) = 45. \]

4. Two 4-sided dice. [Caution: If \( X_j \) is an indicator of whether the minimum of the two dice is “j or greater”—as in the previous homework—then \( X = X_1 + X_2 + X_3 + X_4 \), but the \( X_j \)’s are dependent. So we cannot just sum the variances. We need to find the mass of \( X \) and then compute the expected value and variance by hand.]

The mass of \( X \) is
\[ p_X(1) = 7/16, \quad p_X(2) = 5/16, \quad p_X(3) = 3/16, \quad p_X(4) = 1/16. \]

So
\[ \mathbb{E}(X) = 1(7/16) + 2(5/16) + 3(3/16) + 4(1/16) = 15/8, \]
and
\[ \mathbb{E}(X^2) = 1^2(7/16) + 2^2(5/16) + 3^2(3/16) + 4^2(1/16) = 35/8, \]
and thus
\[ \text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 35/8 - (15/8)^2 = 55/64 = .859375. \]
5. **Pick two cards.** As in Problem Set 7, the mass of $X$ is

\[ p_X(0) = \frac{30}{51}, \quad p_X(1) = \frac{80}{221}, \quad p_X(2) = \frac{11}{221}. \]

So

\[ \mathbb{E}(X) = (0)(\frac{30}{51}) + (1)(\frac{80}{221}) + (2)(\frac{11}{221}) = \frac{6}{13}, \]

and

\[ \mathbb{E}(X^2) = (0^2)(\frac{30}{51}) + (1^2)(\frac{80}{221}) + (2^2)(\frac{11}{221}) = \frac{124}{221}, \]

and thus

\[ \text{Var}(X) = \frac{124}{221} - \left(\frac{6}{13}\right)^2 = \frac{1000}{2873} = .3481. \]

First we find the mass of $Y$. Let $A_1, A_2$ be (respectively) the events that the first, second card is a 10. Even though the cards appear simultaneously, we can just randomly treat one of them as the first and the other as the second. So

\[ P(Y = 0) = P(A_1^c \cap A_2^c) = P(A_1^c)P(A_2^c \mid A_1^c) = (\frac{48}{52})(\frac{47}{51}) = \frac{188}{221}, \]

and

\[ P(Y = 1) = P(A_1 \cap A_2^c) + P(A_1^c \cap A_2) = (\frac{4}{52})(\frac{48}{51}) + (\frac{48}{52})(\frac{4}{51}) = \frac{32}{221}, \]

and

\[ P(Y = 2) = P(A_1 \cap A_2) = (\frac{4}{52})(\frac{3}{51}) = \frac{1}{221}. \]

So

\[ \mathbb{E}(Y) = (0)(\frac{188}{221}) + (1)(\frac{32}{221}) + (2)(\frac{1}{221}) = \frac{2}{13}, \]

and

\[ \mathbb{E}(Y^2) = (0^2)(\frac{188}{221}) + (1^2)(\frac{32}{221}) + (2^2)(\frac{1}{221}) = \frac{36}{221}, \]

and thus

\[ \text{Var}(Y) = \frac{36}{221} - \left(\frac{2}{13}\right)^2 = \frac{400}{2873} = .1392. \]