STAT/MA 41600
Practice Problems: September 22, 2014
Solutions by Mark Daniel Ward

1a. Variance of an Indicator. We have $\mathbb{E}(X^2) = 1^2(p) + 0^2(1-p) = p$. So $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = p - p^2 = p(1-p)$.

1b. Butterflies. *Method #1:* Write X as the sum of three indicator random variables, X_1, X_2, X_3 that indicate whether Alice, Bob, Charlotte (respectively) found a butterfly. Then $X = X_1 + X_2 + X_3$. Since X_1, X_2, X_3 are independent, then $\mathrm{Var}(X) = \mathrm{Var}(X_1 + X_2 + X_3) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \mathrm{Var}(X_3) = (.17)(.83) + (.25)(.75) + (.45)(.55) = .5761$.

Method #2: The mass and expected value of X were given in Problem Set 10:

The mass of X is

$$p_X(0) = 0.3424, \qquad p_X(1) = 0.4644, \qquad p_X(2) = 0.1741, \qquad p_X(3) = 0.0191,$$

so the expected value of X is

$$\mathbb{E}(X) = (0)(0.3424) + (1)(0.4644) + (2)(0.1741) + (3)(0.0191) = .87.$$

The expected value of X^2 is

$$\mathbb{E}(X^2) = (0^2)(0.3424) + (1^2)(0.4644) + (2^2)(0.1741) + (3^2)(0.0191) = 1.33.$$

So the variance of X is $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 1.3327 - (.87)^2 = .58$.

2. Appetizers. *Method #1:* The expected value of X is:

$$\mathbb{E}(X) = (1)(.08) + (2)(.20) + (3)(.32) + (4)(.40) = 3.04,$$

and

$$\mathbb{E}(X^2) = 1^2(.08) + 2^2(.20) + 3^2(.32) + 4^2(.40) = 10.16,$$

so

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 10.16 - (3.04)^2 = .9184.$$

So

$$\mathrm{Var}(Y) = \mathrm{Var}(1.07X) = (1.07)^2\,\mathrm{Var}(X) = (1.07)^2(.9184) = 1.0515.$$

Method #2: The expected value of Y is:

$$\mathbb{E}(Y) = (1.07)(1)(.08) + (1.07)(2)(.20) + (1.07)(3)(.32) + (1.07)(4)(.40) = 3.2528,$$

and

$$\mathbb{E}(Y^2) = ((1.07)(1))^2(.08) + ((1.07)(2))^2(.20) + ((1.07)(3))^2(.32) + ((1.07)(4))^2(.40) = 11.632184,$$

so

$$\mathrm{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 11.632184 - (3.2528)^2 = 1.0515.$$

3. Gloves. As discussed in Problem Set 3, X has mass $1/5$ on the values $1, 2, 3, 4, 5$. So

$$\mathbb{E}(X) = (1)(1/5) + (2)(1/5) + (3)(1/5) + (4)(1/5) + (5)(1/5) = 3.$$

Find $\mathbb{E}(X^2)$.
Again using the mass of X, we have

$$\mathbb{E}(X^2) = 1^2(1/5) + 2^2(1/5) + 3^2(1/5) + 4^2(1/5) + 5^2(1/5) = 11.$$

We have

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 11 - 3^2 = 2.$$

Again using the mass of X, we have

$$\mathbb{E}(X^3) = 1^3(1/5) + 2^3(1/5) + 3^3(1/5) + 4^3(1/5) + 5^3(1/5) = 45.$$

4. Two 4-sided dice. [Caution: If X_j is an indicator of whether the minimum of the two dice is "j or greater"—as in the previous homework—then $X = X_1 + X_2 + X_3 + X_4$, but the X_j's are dependent. So we cannot just sum the variances. We need to find the mass of X and then compute the expected value and variance by hand.]

The mass of X is

$$p_X(1) = 7/16, \qquad p_X(2) = 5/16, \qquad p_X(3) = 3/16, \qquad p_X(4) = 1/16.$$

So

$$\mathbb{E}(X) = 1(7/16) + 2(5/16) + 3(3/16) + 4(1/16) = 15/8,$$

and

$$\mathbb{E}(X^2) = 1^2(7/16) + 2^2(5/16) + 3^2(3/16) + 4^2(1/16) = 35/8,$$

and thus

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 35/8 - (15/8)^2 = 55/64 = .859375.$$

5. Pick two cards. As in Problem Set 7, the mass of X is

$$p_X(0) = 30/51, \qquad p_X(1) = 80/221, \qquad p_X(2) = 11/221.$$

So

$$\mathbb{E}(X) = (0)(30/51) + (1)(80/221) + (2)(11/221) = 6/13,$$

and

$$\mathbb{E}(X^2) = (0^2)(30/51) + (1^2)(80/221) + (2^2)(11/221) = 124/221,$$

and thus

$$\text{Var}(X) = 124/221 - (6/13)^2 = 1000/2873 = .3481.$$

First we find the mass of Y. Let A_1, A_2 be (respectively) the events that the first, second card is a 10. Even though the cards appear simultaneously, we can just randomly treat one of them as the first and the other as the second. So

$$P(Y = 0) = P(A_1^c \cap A_2^c) = P(A_1^c)P(A_2^c \mid A_1^c) = (48/52)(47/51) = 188/221,$$

and

$$P(Y = 1) = P(A_1 \cap A_2^c) + P(A_1^c \cap A_2) = (4/52)(48/51) + (48/52)(4/51) = 32/221,$$

and

$$P(Y = 2) = P(A_1 \cap A_2) = (4/52)(3/51) = 1/221.$$

So

$$\mathbb{E}(Y) = (0)(188/221) + (1)(32/221) + (2)(1/221) = 2/13,$$

and

$$\mathbb{E}(Y^2) = (0^2)(188/221) + (1^2)(32/221) + (2^2)(1/221) = 36/221,$$

and thus

$$\text{Var}(Y) = 36/221 - (2/13)^2 = 400/2873 = .1392.$$