STAT/MA 41600
More Practice Problems: October 6, 2014
Solutions by Mark Daniel Ward

1. Rock Block. Let X be the number of songs in the next "block" that are rock songs. Then

$$P(X = 10 \mid X \geq 8) = \frac{P(X = 10 \text{ and } X \geq 8)}{P(X \geq 8)} = \frac{P(X = 10)}{P(X = 8) + P(X = 9) + P(X = 10)}$$

So the desired probability is $\frac{\binom{10}{10}(.70)^{10}(.30)^0}{\binom{10}{8}(.70)^8(.30)^2+\binom{10}{9}(.70)^9(.30)^1+\binom{10}{10}(.70)^{10}(.30)^0} = \frac{0.02825}{0.23347+0.12106+0.02825} = \frac{0.02825}{0.38278} = 0.07380$.

2. Pizza.

a. There are seven toppings, exactly three of which are needed, and the order is irrelevant, so there are $\binom{7}{3} = 35$ types of 3-topping pizzas.

b. There are 35 types of 3-topping pizzas. If sausage is one of the toppings, then there are $\binom{6}{2} = 15$ ways to pick the other two toppings. So the probability that sausage is a topping is $15/35 = 3/7$.

c. There are 35 types of 3-topping pizzas. If sausage and pepperoni are two of the toppings, then there are $\binom{5}{1} = 5$ ways to pick the other one topping. So the probability that sausage and pepperoni are toppings is $5/35 = 1/7$.

3. Couples in a Circle.

a. It doesn't matter where Alice is seated. Once Alice is seated, there are $3! = 6$ ways that the 3 remaining people can be seated altogether. There are 2 places that Alan can be seated next to Alice (on her right or left). Then there is only 1 remaining pair of seats where the B's can sit. Within that pair of seats, there are 2 ways that the B's can be arranged among themselves (Barbara,Bob or Bob,Barbara). So the total probability that the couples are seated together is $(2)(1)(2)/6 = 2/3$.

b. It doesn't matter where Alice is seated. Once Alice is seated, there are $5! = 120$ ways that the 5 remaining people can be seated altogether. There are 2 places that Alan can be seated next to Alice (on her right or left). Then there are 2! remaining pairs of seats where the B's/C's can sit. For each such way, there are 2 ways that the B's can be arranged among themselves, and 2 ways that the C's can be arranged among themselves. So the total probability that the couples are seated together is $(2)(2!)(2)(2)/120 = 2/15$.

c. It doesn't matter where Alice is seated. Once Alice is seated, there are $(2n-1)!$ ways that the $2n-1$ remaining people can be seated altogether. There are 2 places that Alan can be seated next to Alice (on her right or left). Then there are $(n-1)!$ remaining pairs of seats where the other couples can sit. For each such way, there are 2 ways within *each couple* that the man and woman can be arranged in their two reserved seats. So the total probability that the couples are seated together is $(2)(n-1)!2^{n-1}/(2n-1)!$.

4. Bears. Let X_j be a Bernoulli random variable that indicates, for the jth bowl, whether that bowl has a blue bear. Then $E[X_j] = P(X_j = 1) = 1 - P(X_j = 0) = 1 - \left(\frac{20}{30}\right)\left(\frac{19}{29}\right)\left(\frac{18}{28}\right) = 146/203 \approx .7192$. So $E[X_1 + \cdots + X_{10}] = E[X_1] + \cdots + E[X_{10}] = 10(146/203) \approx 7.192$.

5. More Bears! Let X_j be a Bernoulli random variable that indicates, for the jth bear that is yellow or blue, whether that particular bear is selected before all of the red bears. Thus $E[X_j] = P(X_j = 1) = 1/11$. The total number of bears selected before the first red is $X_1 + \cdots + X_{20}$. Thus $E[X_1 + \cdots + X_{20}] = E[X_1] + \cdots + E[X_{20}] = 1/11 + \cdots + 1/11 = 20/11 \approx 1.818$.