STAT/MA 41600
Practice Problems: October 17, 2014
Solutions by Mark Daniel Ward

1. The joint density is constant on a region of area $(3)(3)/2 = 9/2$. So the joint density $f_X(x)$ is $2/9$ on the triangle, and 0 otherwise.
Method #1: We integrate 2/9 over the region, which is shown in Figure 1(a), and we get

$$\begin{aligned}\int_0^2 \int_{2-x}^{3-x} \frac{2}{9}\,dy\,dx + \int_2^3 \int_0^{3-x} \frac{2}{9}\,dy\,dx &= \int_0^2 \frac{2}{9}y\Big|_{y=2-x}^{3-x}\,dx + \int_2^3 \frac{2}{9}y\Big|_{y=0}^{3-x}\,dx \\ &= \int_0^2 \frac{2}{9}\,dx + \int_2^3 \frac{2}{9}(3-x)\,dx \\ &= \frac{2}{9}x\Big|_{x=0}^{2} + \frac{2}{9}\left(3x - \frac{x^2}{2}\right)\Big|_{x=2}^{3} \\ &= 4/9 + 1/9 \\ &= 5/9\end{aligned}$$

Method #2: We integrate 2/9 over the complementary region, which is shown in Figure 1(b), and we get

$$\begin{aligned}1 - \int_0^2 \int_0^{2-x} \frac{2}{9}\,dy\,dx &= 1 - \int_0^2 \frac{2}{9}y\Big|_{y=0}^{2-x}\,dx \\ &= 1 - \int_0^2 \frac{2}{9}(2-x)\,dx \\ &= 1 - \frac{2}{9}\left(2x - \frac{x^2}{2}\right)\Big|_{x=0}^{2} \\ &= 1 - (2/9)(4-2) \\ &= 1 - 4/9 \\ &= 5/9\end{aligned}$$

Method #3: Actually we don't need to integrate a constant density. We integrate the constant over a region, so the integral is the area of the shaded region (here, 5/2; see Figure 1(a)) over the area of the whole region (here, 9/2), so the probability is $\frac{5/2}{9/2} = 5/9$.

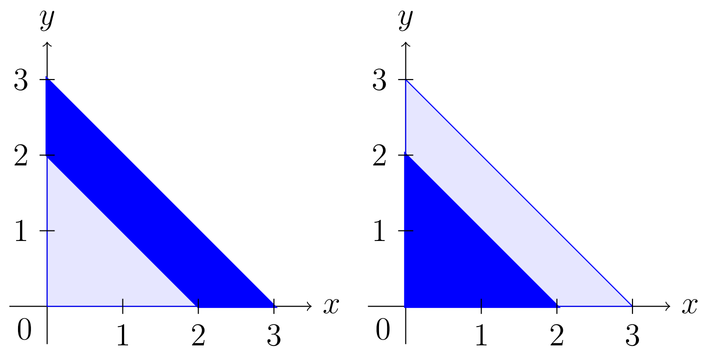

Figure 1: (a.) The region where $X+Y > 2$; (b.) the complementary region, where $X+Y < 2$.

2. The joint density is constant on a region of area 18. So the joint density $f_X(x)$ is 1/18 on the quadrilateral, and 0 otherwise.
Method #1: We integrate 1/18 over the region, which is shown in Figure 2(a), and we get

$$\int_0^2 \int_{3x}^{12-3x} \frac{1}{18}\, dy\, dx = \int_0^2 \frac{1}{18} y \Big|_{y=3x}^{12-3x} dx = \int_0^2 \frac{1}{18}(12-6x)\, dx = \frac{1}{18}\left(12x - 3x^2\right)\Big|_{x=0}^{2} = 2/3.$$

Method #2: We integrate 1/18 over the complementary region, which is shown in Figure 2(b), and we get

$$1 - \int_0^2 \int_0^{3x} \frac{1}{18}\, dy\, dx = 1 - \int_0^2 \frac{1}{18} y \Big|_{y=0}^{3x} dx = 1 - \int_0^2 x/6\, dx = 1 - x^2/12\big|_{x=0}^{2} = 2/3.$$

Method #3: Actually we don't need to integrate a constant density. We integrate the constant over a region, so the integral is the area of the shaded region (here, 12; see Figure 2(a)) over the area of the whole region (here, 18), so the probability is $12/18 = 2/3$.

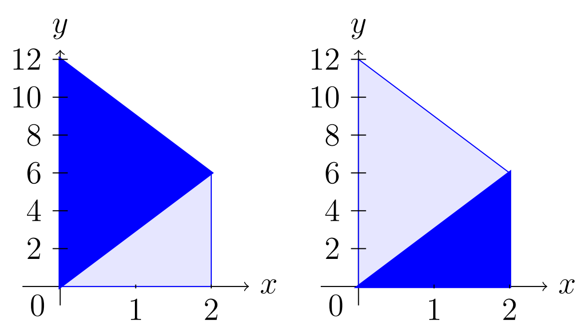

Figure 2: (a.) The region where $Y \geq 3X$; (b.) the complementary region, where $Y \leq 3X$.

3. We have two ways to setup the integral:

Method #1: We can integrate first over all x's (i.e., use integration with respect to x as the outer integral), and then fix x and integrate over all of the y's that are smaller than x, namely, $0 \le y \le x$, as shown in Figure 3.

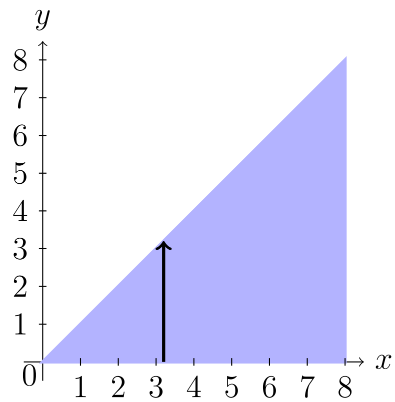

Figure 3: Setting up the integral for $P(X > Y)$, with x as the outer integral and y as the inner integral. Fixed value of x (here, for example $x = 3.2$), and y ranging from 0 to x.

Now we perform the joint integral, as specified in Figure 3, and we get

$$\begin{aligned} P(X > Y) &= \int_0^\infty \int_0^x 14e^{-2x-7y}\,dy\,dx \\ &= \int_0^\infty -2e^{-2x-7y}\Big|_{y=0}^{x}\,dx \\ &= \int_0^\infty (2e^{-2x} - 2e^{-9x})\,dx \\ &= (-e^{-2x} + (2/9)e^{-9x})\Big|_{x=0}^{\infty} \\ &= (1 - (2/9)) \\ &= 7/9 \end{aligned}$$

Method #2: We can integrate first over all y's (i.e., integrating with respect to y as the outer integral), and then fix y and integrate over all of the x's that are larger than y, namely, $y \le x$, as shown in Figure 4.

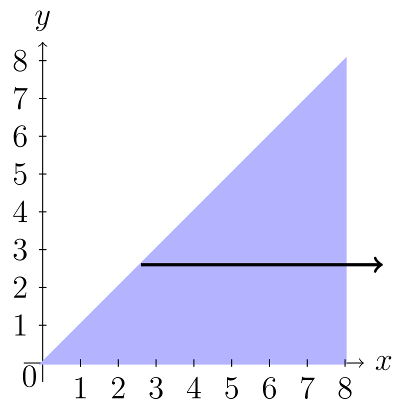

Figure 4: Setting up the integral for $P(X > Y)$, with y as the outer integral and x as the inner integral. Fixed value of y (here, for example $y = 2.6$), and x ranging from y to ∞.

Now we perform the joint integral, as specified in Figure 4, and we get

$$\begin{aligned} P(X > Y) &= \int_0^\infty \int_y^\infty 14e^{-2x-7y}\,dx\,dy \\ &= \int_0^\infty -7e^{-2x-7y}\Big|_{x=y}^{\infty}\,dy \\ &= \int_0^\infty 7e^{-9y}\,dy \\ &= -(7/9)e^{-9y}\Big|_{y=0}^{\infty} \\ &= 7/9 \end{aligned}$$

4. *Method #1:* We can integrate the joint density over the region where $|X - Y| \le 1$, which is shown in Figure 5. The desired probability is

$$\begin{aligned} &\int_{-2}^{-1}\int_{-2}^{x+1} 1/16\,dy\,dx + \int_{-1}^{1}\int_{x-1}^{x+1} 1/16\,dy\,dx + \int_{1}^{2}\int_{x-1}^{2} 1/16\,dy\,dx \\ &\quad= \int_{-2}^{-1}\frac{x+3}{16}\,dx + \int_{-1}^{1}\frac{2}{16}\,dx + \int_{1}^{2}\frac{3-x}{16}\,dx \\ &\quad= \frac{x^2/2+3x}{16}\Big|_{x=-2}^{-1} + \frac{2x}{16}\Big|_{x=-1}^{1} + \frac{3x-x^2/2}{16}\Big|_{x=1}^{2} \\ &= 3/32 + 4/16 + 3/32 \\ &= 14/32 \\ &= 7/16 \end{aligned}$$

Method #2: The desired region has area 7, and the entire region has area 16. Since the joint density is constant, it follows that $P(|X - Y| \le 1) = 7/16$.

Method #3: The complementary region has area 9, and the entire region has area 16. Since the joint density is constant, it follows that $P(|X - Y| \le 1) = 1 - 9/16 = 7/16$.

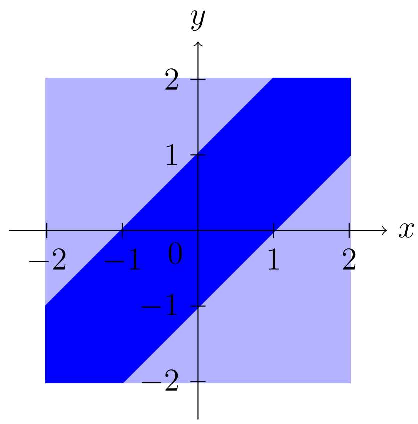

Figure 5: Setting up the integral for $P(|X - Y| \leq 1)$.

5. The region is shown in Figure 6.

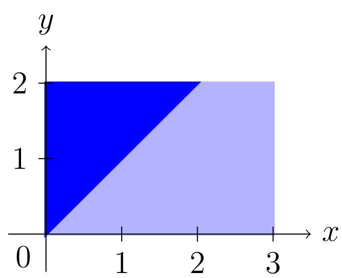

Figure 6: Setting up the integral for $P(Y > X)$.

Method #1: We can integrate with respect to y as the outer integral and with respect to x as the inner integral.

The desired probability is

$$\begin{aligned}
\int_0^2 \int_0^y \frac{1}{9}(3-x)(2-y)\,dx\,dy &= \int_0^2 \frac{1}{9}(3x - x^2/2)(2-y)\Big|_{x=0}^{y}\,dy \\
&= \int_0^2 \frac{1}{9}(3y - y^2/2)(2-y)\,dy \\
&= \int_0^2 \frac{1}{9}(6y - 4y^2 + y^3/2)\,dy \\
&= \frac{1}{9}\left(3y^2 - \frac{4}{3}y^3 + y^4/8\right)\Big|_{y=0}^{2} \\
&= \frac{1}{9}\left(3(2)^2 - \frac{4}{3}(2)^3 + (2)^4/8\right) \\
&= (1/9)(12 - 32/3 + 2) \\
&= 10/27
\end{aligned}$$

Method #2: We can integrate with respect to x as the outer integral and with respect to y as the inner integral.

The desired probability is

$$\begin{aligned}
\int_0^2 \int_x^2 \frac{1}{9}(3-x)(2-y)\,dy\,dx &= \int_0^2 \frac{1}{9}(3-x)(2y - y^2/2)\Big|_{y=x}^{2}\,dx \\
&= \int_0^2 \frac{1}{9}(3-x)(2 - 2x + x^2/2)\,dx \\
&= \int_0^2 \frac{1}{9}\left(6 - 8x + \frac{7}{2}x^2 - x^3/2\right)\,dx \\
&= \frac{1}{9}\left(6x - 4x^2 + \frac{7}{6}x^3 - x^4/8\right)\Big|_{x=0}^{2} \\
&= \frac{1}{9}\left(6(2) - 3(2)^2 + \frac{1}{2}(2)^3 - (2)^2 + \frac{2}{3}(2)^3 - (2)^4/8\right) \\
&= 10/27
\end{aligned}$$