

STAT/MA 41600  
Practice Problems: October 20, 2014  
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1.

a. No,  $X$  and  $Y$  are not independent. They are dependent. This is perhaps easiest to see because they are not defined on rectangles. So, for instance,  $P(X > 2 \text{ and } Y > 2) = 0$  but  $P(X > 2)P(Y > 2) \neq 0$ .

b. The density of  $f_X(x)$ , for  $0 \leq x \leq 3$ , is  $f_X(x) = \int_0^{3-x} 2/9 dy = \frac{2}{9}y \Big|_{y=0}^{3-x} = \frac{2}{9}(3-x)$ . So

$$f_X(x) = \begin{cases} \frac{2}{9}(3-x) & \text{if } 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

c. The density of  $f_Y(y)$ , for  $0 \leq y \leq 3$ , is  $f_Y(y) = \int_0^{3-y} 2/9 dx = \frac{2}{9}x \Big|_{x=0}^{3-y} = \frac{2}{9}(3-y)$ . So

$$f_Y(y) = \begin{cases} \frac{2}{9}(3-y) & \text{if } 0 \leq y \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

2.

a. No,  $X$  and  $Y$  are not independent. They are dependent. This is again perhaps easiest to see because they are not defined on rectangles. So, for instance,  $P(X > 1 \text{ and } Y > 11) = 0$  but  $P(X > 1)P(Y > 11) \neq 0$ .

b. The density of  $f_X(x)$ , for  $0 \leq x \leq 2$ , is  $f_X(x) = \int_0^{12-3x} 1/18 dy = \frac{1}{18}y \Big|_{y=0}^{12-3x} = \frac{1}{18}(12-3x) = \frac{1}{6}(4-x)$ . So

$$f_X(x) = \begin{cases} \frac{1}{6}(4-x) & \text{if } 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

c. The density of  $f_Y(y)$ , for  $0 \leq y \leq 6$ , is  $f_Y(y) = \int_0^2 1/18 dx = 1/9$ .

The density of  $f_Y(y)$ , for  $6 \leq y \leq 12$ , is  $f_Y(y) = \int_0^{(12-y)/3} 1/18 dx = \frac{1}{18}x \Big|_{x=0}^{(12-y)/3} = \frac{1}{18}(12-y)/3 = (12-y)/54$ . So

$$f_Y(y) = \begin{cases} \frac{1}{9} & \text{if } 0 \leq y \leq 6 \\ \frac{12-y}{54} & \text{if } 6 \leq y \leq 12 \\ 0 & \text{otherwise} \end{cases}$$

3.

a. Yes,  $X$  and  $Y$  are independent, because their joint density  $f_{X,Y}(x,y)$  can be factored into the  $x$  stuff times the  $y$  stuff, e.g., we can write  $14e^{-2x-7y} = 14e^{-2x}e^{-7y}$ .

b. The density of  $f_X(x)$ , for  $x > 0$ , is  $f_X(x) = \int_0^\infty 14e^{-2x-7y} dy = -2e^{-2x-7y} \Big|_{y=0}^\infty = 2e^{-2x}$ . So

$$f_X(x) = \begin{cases} 2e^{-2x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

c. The density of  $f_Y(y)$ , for  $y > 0$ , is  $f_Y(y) = \int_0^\infty 14e^{-2x-7y} dx = -7e^{-2x-7y} \Big|_{x=0}^\infty = 7e^{-7y}$ .  
So

$$f_Y(y) = \begin{cases} 7e^{-7y} & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

4.

a. Yes,  $X$  and  $Y$  are independent, because their joint density  $f_{X,Y}(x,y)$  can be factored into the  $x$  stuff times the  $y$  stuff, e.g., we can right  $1/16 = (1/4)(1/4)$ , and these are the densities of  $X$  and  $Y$ , as we will see below.

b. The density of  $f_X(x)$ , for  $-2 \leq x \leq 2$ , is  $f_X(x) = \int_{-2}^2 1/16 dy = 1/4$ . So

$$f_X(x) = \begin{cases} 1/4 & \text{if } -2 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

c. The density of  $f_Y(y)$ , for  $-2 \leq y \leq 2$ , is  $f_Y(y) = \int_{-2}^2 1/16 dx = 1/4$ . So

$$f_Y(y) = \begin{cases} 1/4 & \text{if } -2 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

5.

a. Yes,  $X$  and  $Y$  are independent, because their joint density  $f_{X,Y}(x,y) = \frac{1}{9}(3-x)(2-y)$  is already factored into the  $x$  stuff times the  $y$  stuff.

b. The density of  $f_X(x)$ , for  $0 \leq x \leq 3$ , is  $f_X(x) = \int_0^2 \frac{1}{9}(3-x)(2-y) dy = \frac{1}{9}(3-x) \left( 2y - \frac{y^2}{2} \right) \Big|_{y=0}^2 = \frac{1}{9}(3-x) \left( 2(2) - \frac{2^2}{2} \right) = \frac{2}{9}(3-x)$ . So

$$f_X(x) = \begin{cases} \frac{2}{9}(3-x) & \text{if } 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

c. The density of  $f_Y(y)$ , for  $0 \leq y \leq 2$ , is  $f_Y(y) = \int_0^3 \frac{1}{9}(3-x)(2-y) dx = \frac{1}{9} \left( 3x - \frac{x^2}{2} \right) (2-y) \Big|_{x=0}^3 = \frac{1}{9} \left( 3(3) - \frac{3^2}{2} \right) (2-y) = \frac{1}{2}(2-y)$ . So

$$f_Y(y) = \begin{cases} \frac{1}{2}(2-y) & \text{if } 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$