STAT/MA 41600
Practice Problems: October 20, 2014
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1.

a. No, X and Y are not independent. They are dependent. This is perhaps easiest to see because they are not defined on rectangles. So, for instance, $P(X > 2 \text{ and } Y > 2) = 0$ but $P(X > 2)P(Y > 2) \neq 0$.

b. The density of $f_X(x)$, for $0 \leq x \leq 3$, is $f_X(x) = \int_0^{3-x} 2/9\,dy = \frac{2}{9}y\big|_{y=0}^{3-x} = \frac{2}{9}(3-x)$. So

$$f_X(x) = \begin{cases} \frac{2}{9}(3-x) & \text{if } 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

c. The density of $f_Y(y)$, for $0 \leq y \leq 3$, is $f_Y(y) = \int_0^{3-y} 2/9\,dx = \frac{2}{9}x\big|_{x=0}^{3-y} = \frac{2}{9}(3-y)$. So

$$f_Y(y) = \begin{cases} \frac{2}{9}(3-y) & \text{if } 0 \leq y \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

2.

a. No, X and Y are not independent. They are dependent. This is again perhaps easiest to see because they are not defined on rectangles. So, for instance, $P(X > 1 \text{ and } Y > 11) = 0$ but $P(X > 1)P(Y > 11) \neq 0$.

b. The density of $f_X(x)$, for $0 \leq x \leq 2$, is $f_X(x) = \int_0^{12-3x} 1/18\,dy = \frac{1}{18}y\big|_{y=0}^{12-3x} = \frac{1}{18}(12-3x) = \frac{1}{6}(4-x)$. So

$$f_X(x) = \begin{cases} \frac{1}{6}(4-x) & \text{if } 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

c. The density of $f_Y(y)$, for $0 \leq y \leq 6$, is $f_Y(y) = \int_0^2 1/18\,dx = 1/9$.

The density of $f_Y(y)$, for $6 \leq y \leq 12$, is $f_Y(y) = \int_0^{(12-y)/3} 1/18\,dx = \frac{1}{18}x\big|_{x=0}^{(12-y)/3} = \frac{1}{18}(12-y)/3 = (12-y)/54$. So

$$f_Y(y) = \begin{cases} \frac{1}{9} & \text{if } 0 \leq y \leq 6 \\ \frac{12-y}{54} & \text{if } 6 \leq y \leq 12 \\ 0 & \text{otherwise} \end{cases}$$

3.

a. Yes, X and Y are independent, because their joint density $f_{X,Y}(x,y)$ can be factored into the x stuff times the y stuff, e.g., we can right $14e^{-2x-7y} = 14e^{-2x}e^{-7y}$.

b. The density of $f_X(x)$, for $x > 0$, is $f_X(x) = \int_0^\infty 14e^{-2x-7y}\,dy = -2e^{-2x-7y}\big|_{y=0}^{\infty} = 2e^{-2x}$. So

$$f_X(x) = \begin{cases} 2e^{-2x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

c. The density of $f_Y(y)$, for $y > 0$, is $f_Y(y) = \int_0^\infty 14e^{-2x-7y}\,dx = -7e^{-2x-7y}\big|_{x=0}^\infty = 7e^{-7y}$. So

$$f_Y(y) = \begin{cases} 7e^{-7y} & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

4.

a. Yes, X and Y are independent, because their joint density $f_{X,Y}(x,y)$ can be factored into the x stuff times the y stuff, e.g., we can right $1/16 = (1/4)(1/4)$, and these are the densities of X and Y, as we will see below.

b. The density of $f_X(x)$, for $-2 \le x \le 2$, is $f_X(x) = \int_{-2}^2 1/16\,dy = 1/4$. So

$$f_X(x) = \begin{cases} 1/4 & \text{if } -2 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

c. The density of $f_Y(y)$, for $-2 \le y \le 2$, is $f_Y(y) = \int_{-2}^2 1/16\,dx = 1/4$. So

$$f_Y(y) = \begin{cases} 1/4 & \text{if } -2 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

5.

a. Yes, X and Y are independent, because their joint density $f_{X,Y}(x,y) = \frac{1}{9}(3-x)(2-y)$ is already factored into the x stuff times the y stuff.

b. The density of $f_X(x)$, for $0 \le x \le 3$, is $f_X(x) = \int_0^2 \frac{1}{9}(3-x)(2-y)\,dy = \frac{1}{9}(3-x)\left(2y - \frac{y^2}{2}\right)\Big|_{y=0}^2 = \frac{1}{9}(3-x)\left(2(2) - \frac{2^2}{2}\right) = \frac{2}{9}(3-x)$. So

$$f_X(x) = \begin{cases} \frac{2}{9}(3-x) & \text{if } 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

c. The density of $f_Y(y)$, for $0 \le y \le 2$, is $f_Y(y) = \int_0^3 \frac{1}{9}(3-x)(2-y)\,dx = \frac{1}{9}\left(3x - \frac{x^2}{2}\right)(2-y)\Big|_{x=0}^3 = \frac{1}{9}\left(3(3) - \frac{3^2}{2}\right)(2-y) = \frac{1}{2}(2-y)$. So

$$f_Y(y) = \begin{cases} \frac{1}{2}(2-y) & \text{if } 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$