

STAT/MA 41600
Practice Problems: October 22, 2014
Solutions by Mark Daniel Ward

1.

a. The density of Y , for $0 \leq y \leq 3$, is $f_Y(y) = \frac{2}{9}(3 - y)$, as we saw in the previous problem set. The joint density is $2/9$. Thus, the conditional density of X given Y is

$$f_{X|Y}(x | y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \frac{2/9}{\frac{2}{9}(3 - y)} = \frac{1}{3 - y}, \quad \text{for } 0 \leq x \leq 3 - y,$$

and $f_{X|Y}(x | y) = 0$ otherwise.

b. The conditional probability is $P(X \leq 1 | Y = 1) = \int_0^1 f_{X|Y}(x | 1) dx = \int_0^1 \frac{1}{3-1} dx = 1/2$.

c. Using Bayes' Theorem, we have $P(X \leq 1 | Y \leq 1) = \frac{P(X \leq 1 \text{ and } Y \leq 1)}{P(Y \leq 1)} = \frac{1/(9/2)}{(5/2)/(9/2)} = 2/5$.

2.

a. The density of Y , for $0 \leq y \leq 6$, is $f_Y(y) = 1/9$, as we saw in the previous problem set. The joint density is $1/18$. Thus, the conditional density of X given Y is

$$f_{X|Y}(x | y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \frac{1/18}{1/9} = 1/2, \quad \text{for } 0 \leq x \leq 2,$$

and $f_{X|Y}(x | y) = 0$ otherwise.

b. The density of Y , for $6 \leq y \leq 12$, is $f_Y(y) = (12 - y)/54$, as we saw in the previous problem set. The joint density is $1/18$. Thus, the conditional density of X given Y is

$$f_{X|Y}(x | y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \frac{1/18}{(12 - y)/54} = 3/(12 - y), \quad \text{for } 0 \leq x \leq (12 - y)/3,$$

and $f_{X|Y}(x | y) = 0$ otherwise.

c. Using Bayes' Theorem, $P(X \leq 1 | 3 \leq Y \leq 9) = \frac{P(X \leq 1 \text{ and } 3 \leq Y \leq 9)}{P(3 \leq Y \leq 9)} = \frac{6/18}{10.5/18} = 4/7$.

3.

a. Since X and Y are independent, then $f_{X|Y}(x | y) = f_X(x)$. Thus $f_{X|Y}(x | y) = 2e^{-2x}$ for $x > 0$, and $f_{X|Y}(x | y) = 0$ otherwise.

b. Since X and Y are independent, then $P(X \geq 1 | Y = 3) = P(X \geq 1) = \int_1^\infty 2e^{-2x} dx = -e^{-2x} \Big|_{x=1}^\infty = e^{-2} = 0.1353$.

c. Since X and Y are independent, then $f_{Y|X}(y | x) = f_Y(y)$. Thus $f_{Y|X}(y | x) = 7e^{-7y}$ for $y > 0$, and $f_{Y|X}(y | x) = 0$ otherwise. So $P(Y \leq 1/5 | X = 2.7) = P(Y \leq 1/5) = \int_0^{1/5} 7e^{-7y} dy = -e^{-7y} \Big|_{y=0}^{1/5} = 1 - e^{-7/5} = 0.7534$.

4.

a. The density of $f_Y(y)$, for $y > 0$, is $f_Y(y) = \int_y^\infty 18e^{-2x-7y} dx = -9e^{-2x-7y} \Big|_{x=y}^\infty = 9e^{-9y}$. So

$$f_{X|Y}(x | y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \frac{18e^{-2x-7y}}{9e^{-9y}} = 2e^{-2x+2y} \quad \text{if } x > y,$$

and $f_{X|Y}(x | y) = 0$ otherwise.

b. The density of $f_X(x)$, for $x > 0$, is $f_X(x) = \int_0^x 18e^{-2x-7y} dy = \frac{18}{7} (e^{-2x} - e^{-9x})$. So

$$f_{Y|X}(y | x) = \frac{f_{X,Y}(x, y)}{f_X(x)} = \frac{18e^{-2x-7y}}{\frac{18}{7}(e^{-2x} - e^{-9x})} = \frac{7e^{-7y}}{(1 - e^{-7x})} \quad \text{if } 0 < y < x,$$

and $f_{Y|X}(y | x) = 0$ otherwise.

5.

a. Since X and Y are independent, then $f_{X|Y}(x | y) = f_X(x)$. Thus $f_{X|Y}(x | y) = \frac{2}{9}(3-x)$ for $0 \leq x \leq 3$, and $f_{X|Y}(x | y) = 0$ otherwise.

b. Since X and Y are independent, $P(X \leq 2 | Y = 3/2) = P(X \leq 2) = \int_0^2 \frac{2}{9}(3-x) dx = \frac{2}{9} \left(3x - \frac{x^2}{2}\right) \Big|_{x=0}^2 = 8/9$.

c. Since X and Y are independent, $P(Y \geq 1 | X = 5/4) = P(Y \geq 1) = \int_1^2 \frac{1}{2}(2-y) dy = \frac{1}{2} \left(2y - \frac{y^2}{2}\right) \Big|_{y=1}^2 = 1/4$.