

STAT/MA 41600  
Practice Problems: October 24, 2014  
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1. As we saw earlier, the density of  $X$  is

$$f_X(x) = \begin{cases} \frac{2}{9}(3-x) & \text{if } 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Thus } \mathbb{E}(X) = \int_0^3 \frac{2}{9}(3-x)x \, dx = \frac{2}{9} \int_0^3 (3x - x^2) \, dx = \frac{2}{9} \left( \frac{3x^2}{2} - \frac{x^3}{3} \right) \Big|_{x=0}^3 = \frac{2}{9} \left( \frac{27}{2} - \frac{27}{3} \right) = 1.$$

2.

a. As we saw earlier, the density of  $X$  is

$$f_X(x) = \begin{cases} \frac{1}{6}(4-x) & \text{if } 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Thus } \mathbb{E}(X) = \int_0^2 \frac{1}{6}(4-x)x \, dx = \frac{1}{6} \int_0^2 (4x - x^2) \, dx = \frac{1}{6} \left( \frac{4x^2}{2} - \frac{x^3}{3} \right) \Big|_{x=0}^2 = \frac{1}{6} \left( 8 - \frac{8}{3} \right) = 8/9.$$

b. As we saw earlier, the density of  $Y$  is

$$f_Y(y) = \begin{cases} \frac{1}{9} & \text{if } 0 \leq y \leq 6 \\ \frac{12-y}{54} & \text{if } 6 \leq y \leq 12 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Thus } \mathbb{E}(Y) = \int_0^6 \frac{1}{9}y \, dy + \int_6^{12} \frac{12-y}{54}y \, dy = \frac{1}{9} \frac{y^2}{2} \Big|_{y=0}^6 + \int_6^{12} \frac{12y-y^2}{54} \, dy = \frac{1}{9}(18) + \frac{1}{54} \left( 6y^2 - \frac{y^3}{3} \right) \Big|_{y=6}^{12} = 2 + \frac{1}{54} ((864 - 576) - (216 - 72)) = 14/3.$$

3.

a. As we saw earlier, the density of  $X$  is

$$f_X(x) = \begin{cases} 2e^{-2x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Thus, using } u = x \text{ and } dv = 2e^{-2x} \text{ in integration by parts, we have } du = dx \text{ and } v = -e^{-2x}, \text{ so we get } \mathbb{E}(X) = \int_0^\infty 2e^{-2x}x \, dx = -xe^{-2x} \Big|_{x=0}^\infty - \int_0^\infty -e^{-2x} \, dx = \frac{e^{-2x}}{-2} \Big|_{x=0}^\infty = \frac{1}{2}.$$

b. As we saw earlier, the density of  $Y$  is

$$f_Y(y) = \begin{cases} 7e^{-7y} & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Thus, using } u = y \text{ and } dv = 7e^{-7y} \text{ in integration by parts, we have } du = dy \text{ and } v = -e^{-7y}, \text{ so we get } \mathbb{E}(Y) = \int_0^\infty 7e^{-7y}y \, dy = -ye^{-7y} \Big|_{y=0}^\infty - \int_0^\infty -e^{-7y} \, dy = \frac{e^{-7y}}{-7} \Big|_{y=0}^\infty = \frac{1}{7}.$$

4.

a. As we saw earlier, the density of  $X$  is

$$f_X(x) = \begin{cases} \frac{18}{7}(e^{-2x} - e^{-9x}) & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus,  $\mathbb{E}(X) = \int_0^\infty \frac{18}{7}(e^{-2x} - e^{-9x})x dx = \frac{18}{7}(\int_0^\infty e^{-2x}x dx - \int_0^\infty e^{-9x}x dx)$ . Using  $u = x$  and  $dv = e^{-2x}$  in integration by parts, we have  $du = dx$  and  $v = -e^{-2x}/2$ , so  $\int_0^\infty e^{-2x}x dx = -xe^{-2x}/2|_{x=0}^\infty - \int_0^\infty -e^{-2x}/2 dx = \frac{e^{-2x}}{-4}|_{x=0}^\infty = 1/4$ . Similarly, using  $u = x$  and  $dv = e^{-9x}$  in integration by parts, we have  $du = dx$  and  $v = -e^{-9x}/9$ , so  $\int_0^\infty e^{-9x}x dx = -xe^{-9x}/9|_{x=0}^\infty - \int_0^\infty -e^{-9x}/9 dx = \frac{e^{-9x}}{-81}|_{x=0}^\infty = 1/81$ . Thus  $\mathbb{E}(X) = \frac{18}{7}(\frac{1}{4} - \frac{1}{81}) = 11/18$ .

b. As we saw earlier, the density of  $Y$  is

$$f_Y(y) = \begin{cases} 9e^{-9y} & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, using  $u = y$  and  $dv = 9e^{-9y}$  in integration by parts, we have  $du = dy$  and  $v = -e^{-9y}$ , so we get  $\mathbb{E}(Y) = \int_0^\infty 9e^{-9y}y dy = -ye^{-9y}|_{y=0}^\infty - \int_0^\infty -e^{-9y} dy = \frac{e^{-9y}}{-9}|_{y=0}^\infty = \frac{1}{9}$ .

5.

a. As we saw earlier, the density of  $X$  is

$$f_X(x) = \begin{cases} \frac{2}{9}(3-x) & \text{if } 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

So, exactly as in Question 1, we have:

$$\mathbb{E}(X) = \int_0^3 \frac{2}{9}(3-x)x dx = \frac{2}{9} \int_0^3 (3x - x^2) dx = \frac{2}{9} \left( \frac{3x^2}{2} - \frac{x^3}{3} \right) \Big|_{x=0}^3 = \frac{2}{9} \left( \frac{27}{2} - 9 \right) = 1.$$

b. As we saw earlier, the density of  $Y$  is

$$f_Y(y) = \begin{cases} \frac{1}{2}(2-y) & \text{if } 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Thus } \mathbb{E}(Y) = \int_0^2 \frac{1}{2}(2-y)y dy = \frac{1}{2} \int_0^2 (2y - y^2) dy = \frac{1}{2} \left( y^2 - \frac{y^3}{3} \right) \Big|_{y=0}^2 = \frac{1}{2} \left( 4 - \frac{8}{3} \right) = 2/3.$$