STAT/MA 41600
Practice Problems: October 29, 2014
Solutions by Mark Daniel Ward

1.

a. The probability is $P(X \leq 4.5) = F_X(4.5) = \frac{4.5-2}{4} = 0.625$.

b. *Method #1:* The probability is $P(3.09 \leq X \leq 4.39) = P(X \leq 4.39) - P(X < 3.09) = F_X(4.39) - F_X(3.09) = \frac{4.39-2}{4} - \frac{3.09-2}{4} = 0.325$.

Method #2: The density is $f_X(x) = \frac{d}{dx}F_X(x) = 1/4$ for $2 \leq x \leq 6$ and $f_X(x) = 0$ otherwise. So $P(3.09 \leq X \leq 4.39) = \int_{3.09}^{4.39} 1/4\,dx = 1.3/4 = 0.325$.

Method #3: Since X has continuous uniform distribution, we can use the lengths of the line segments, to compute $P(3.09 \leq X \leq 4.39) = \frac{\text{length of } [3.09,4.39]}{\text{length of } [2,6]} = \frac{1.3}{4} = 0.325$.

c. The probability is $P(X \geq 3.7) = 1 - P(X < 3.7) = 1 - F_X(3.7) = 1 - \frac{3.7-2}{4} = 0.575$.

2.

a. *Method #1:* Using the CDF formula, the probability is $P(X > 12) = 1 - P(X \leq 12) = 1 - F_X(12) = 1 - \frac{12-11.93}{12.02-11.93} = 0.222$.

Method #2: The density is $f_X(x) = \frac{1}{12.02-11.93} = \frac{1}{0.09}$ for $11.93 \leq x \leq 12.02$ and $f_X(x) = 0$ otherwise. So $P(X \geq 12) = \int_{12}^{12.02} \frac{1}{0.09}\,dx = \frac{0.02}{0.09} = 0.222$.

Method #3: Since X has continuous uniform distribution, we can use the lengths of the line segments, to compute $P(X \geq 12) = \frac{\text{length of } [12,12.02]}{\text{length of } [11.93,12.02]} = \frac{0.02}{0.09} = 0.222$.

b. Since the amount of soda is uniform on the interval $[11.93, 12.02]$, then the variance is $(12.02 - 11.93)^2/12 = 0.000675$, so the standard deviation is $\sqrt{0.000675} = 0.02598$ ounces.

3.

a. We write X as the quantity of gasoline, so that X is uniform on $[4.30, 4.50]$ and the cost of the purchase is $12X + 1.00$. So $\mathbb{E}(X) = (4.30 + 4.50)/2 = 4.40$, and thus the expected value of the cost of the purchase is $12(4.40) + 1.00 = 53.80$ dollars.

b. Using the notation from part (a), we have $\text{Var}(X) = (4.50 - 4.30)^2/12 = 0.003333$. Thus, the variance of the purchase cost is $\text{Var}(12X + 1.00) = 12^2\,\text{Var}(X) = (144)(0.003333) = 0.48$.

4. *Method #1:* The three random variables X, Y, Z are independent and identically distributed, so any of the three of them is equally-likely to be the middle value. Thus Y is the middle value with probability 1/3.

Method #2: Each of the random variables has density 1/10, so the joint density is $f_{X,Y,Z}(x, y, z) = 1/1000$. Thus, we can integrate

$$P(X < Y < Z) = \int_0^{10}\int_0^z\int_0^y \frac{1}{1000}\,dx\,dy\,dz = \int_0^{10}\int_0^z \frac{y}{1000}\,dy\,dz = \int_0^{10} \frac{z^2/2}{1000}\,dz = \frac{10^3/6}{1000} = 1/6,$$

and

$$P(Z < Y < X) = \int_0^{10}\int_0^x\int_0^y \frac{1}{1000}\,dz\,dy\,dx = \int_0^{10}\int_0^x \frac{y}{1000}\,dy\,dx = \int_0^{10} \frac{x^2/2}{1000}\,dx = \frac{10^3/6}{1000} = 1/6,$$

so we add the probabilities of these disjoint events: $P(X < Y < Z \text{ or } Z < Y < X) = P(X < Y < Z) + P(Z < Y < X) = 1/6 + 1/6 = 1/3$.

5. We use Figure 1 to guide the way to setup the integral. The joint density of X and Y, as we have seen in previous problem sets, is $f_{X,Y}(x,y) = 2/9$ for X, Y in the triangle, and $f_{X,Y}(x,y) = 0$ otherwise. So we have

$$\begin{aligned}
\mathbb{E}(\min(X,Y)) &= \int_0^{3/2}\int_0^x \frac{2}{9} y\,dy\,dx + \int_{3/2}^3\int_0^{3-x} \frac{2}{9} y\,dy\,dx + \int_0^{3/2}\int_0^y \frac{2}{9} x\,dx\,dy + \int_{3/2}^3\int_0^{3-y} \frac{2}{9} x\,dx\,dy \\
&= \int_0^{3/2} \frac{x^2}{9}\,dx + \int_{3/2}^3 \frac{(3-x)^2}{9}\,dx + \int_0^{3/2} \frac{y^2}{9}\,dy + \int_{3/2}^3 \frac{(3-y)^2}{9}\,dy \\
&= 1/8 + 1/8 + 1/8 + 1/8 \\
&= 1/2
\end{aligned}$$

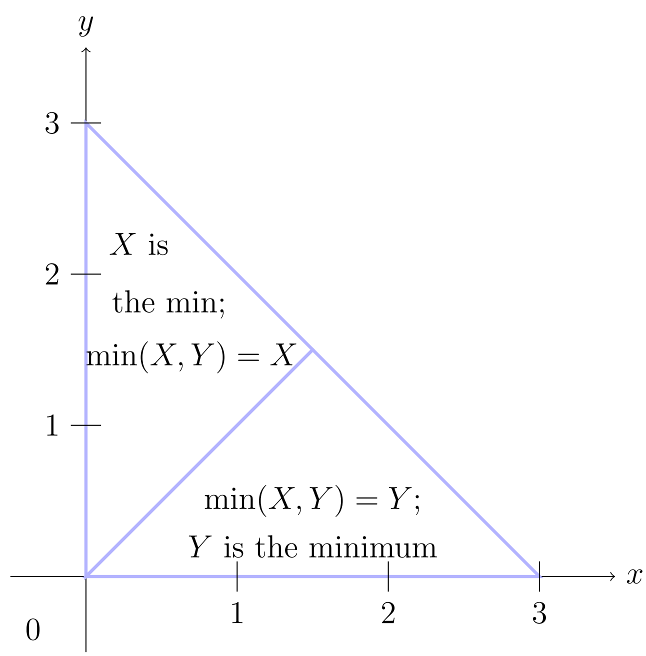

Figure 1: The regions where $\min(X,Y) = X$ versus where $\min(X,Y) = Y$.