STAT/MA 41600
Practice Problems: October 29, 2014
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1. a. The probability is \( P(X \leq 4.5) = F_X(4.5) = \frac{4.5-2}{4} = 0.625 \).

b. Method #1: The probability is \( P(3.09 \leq X \leq 4.39) = P(X \leq 4.39) - P(X < 3.09) = F_X(4.39) - F_X(3.09) = \frac{4.39-2}{4} - \frac{3.09-2}{4} = 0.325 \).

Method #2: The density is \( f_X(x) = \frac{d}{dx} F_X(x) = 1/4 \) for \( 2 \leq x \leq 6 \) and \( f_X(x) = 0 \) otherwise. So \( P(3.09 \leq X \leq 4.39) = \int_{3.09}^{4.39} 1/4 \, dx = 1.3/4 = 0.325 \).

Method #3: Since \( X \) has continuous uniform distribution, we can use the lengths of the line segments, to compute \( P(3.09 \leq X \leq 4.39) = \frac{\text{length of } [3.09, 4.39]}{\text{length of } [2, 6]} = \frac{1.3}{4} = 0.325 \).

c. The probability is \( P(X \geq 3.7) = 1 - P(X < 3.7) = 1 - F_X(3.7) = 1 - \frac{3.7-2}{4} = 0.575 \).

2. a. Method #1: Using the CDF formula, the probability is \( P(X > 12) = 1 - P(X \leq 12) = 1 - F_X(12) = 1 - \frac{12 - 11.93}{12.02 - 11.93} = 0.222 \).

Method #2: The density is \( f_X(x) = \frac{1}{12.02 - 11.93} \) for \( 11.93 \leq x \leq 12.02 \) and \( f_X(x) = 0 \) otherwise. So \( P(X \geq 12) = \int_{12}^{12.02} \frac{1}{0.09} \, dx = 0.92 = 0.222 \).

Method #3: Since \( X \) has continuous uniform distribution, we can use the lengths of the line segments, to compute \( P(X \geq 12) = \frac{\text{length of } [11.93, 12.02]}{\text{length of } [11.93, 12.02]} = \frac{0.02}{0.09} = 0.222 \).

b. Since the amount of soda is uniform on the interval \([11.93, 12.02]\), then the variance is \((12.02 - 11.93)^2/12 = 0.000675\), so the standard deviation is \( \sqrt{0.000675} = 0.02598 \) ounces.

3. a. We write \( X \) as the quantity of gasoline, so that \( X \) is uniform on \([4.30, 4.50]\) and the cost of the purchase is \( 12X + 1.00 \). So \( \mathbb{E}(X) = (4.30 + 4.50)/2 = 4.40 \), and thus the expected value of the cost of the purchase is \( 12(4.40) + 1.00 = 53.80 \) dollars.

b. Using the notation from part (a), we have \( \text{Var}(X) = (4.50 - 4.30)^2/12 = 0.003333 \). Thus, the variance of the purchase cost is \( \text{Var}(12X + 1.00) = 12^2 \text{Var}(X) = (144)(0.003333) = 0.48 \).

4. Method #1: The three random variables \( X, Y, Z \) are independent and identically distributed, so any of the three of them is equally-likely to be the middle value. Thus \( Y \) is the middle value with probability \( 1/3 \).

Method #2: Each of the random variables has density \( 1/10 \), so the joint density is \( f_{X,Y,Z}(x, y, z) = 1/1000 \). Thus, we can integrate

\[
P(X < Y < Z) = \int_0^{10} \int_0^y \frac{1}{1000} \, dx \, dy \, dz = \int_0^{10} \int_0^z \frac{y}{1000} \, dy \, dz = \int_0^{10} \frac{z^2/2}{1000} \, dz = \frac{10^3/6}{1000} = 1/6,
\]
and
\[ P(Z < Y < X) = \int_0^1 \int_0^y \frac{1}{1000} \, dz \, dy \, dx = \int_0^1 \int_0^x \frac{y}{1000} \, dy \, dx = \int_0^{10} \frac{x^{3/2}}{1000} \, dx = \frac{10^{3/6}}{1000} = 1/6, \]
so we add the probabilities of these disjoint events: \( P(X < Y < Z \text{ or } Z < Y < X) = P(X < Y < Z) + P(Z < Y < X) = 1/6 + 1/6 = 1/3. \)

5. We use Figure 1 to guide the way to setup the integral. The joint density of \( X \) and \( Y \), as we have seen in previous problem sets, is \( f_{X,Y}(x,y) = 2/9 \) for \( X, Y \) in the triangle, and \( f_{X,Y}(x,y) = 0 \) otherwise. So we have
\[
E(\min(X,Y)) = \int_0^{3/2} \int_0^x \frac{2}{9} \, dy \, dx + \int_0^{3/2} \int_0^{3-x} \frac{2}{9} \, dy \, dx + \int_0^{3/2} \int_0^y \frac{2}{9} \, dx \, dy + \int_0^{3/2} \int_0^{3-y} \frac{2}{9} \, x \, dx \, dy
\]
\[ = \int_0^{3/2} \frac{x^2}{9} \, dx + \int_0^{3/2} \frac{(3-x)^2}{9} \, dx + \int_0^{3/2} \frac{y^2}{9} \, dy + \int_0^{3/2} \frac{(3-y)^2}{9} \, dy
\]
\[ = 1/8 + 1/8 + 1/8 + 1/8
\]
\[ = 1/2
\]

Figure 1: The regions where \( \min(X,Y) = X \) versus where \( \min(X,Y) = Y \).