STAT/MA 41600
Practice Problems: November 5, 2014
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1. a. *Method #1:* Since Y is a Gamma random variable with $1/\lambda = 30$ and $r = 3$, then $\mathbb{E}(Y) = r/\lambda = 90$ minutes.

Method #2: We can just add the expected values: $\mathbb{E}(Y) = \mathbb{E}(X_1 + X_2 + X_3) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) = 30 + 30 + 30 = 90$.

b. *Method #1:* Since Y is a Gamma random variable with $1/\lambda = 30$ and $r = 3$, then $\mathrm{Var}(Y) = r/\lambda^2 = 2700$, so $\sigma_Y = \sqrt{2700} = 51.96$ minutes.

Method #2: Since X_1, X_2, X_3 are independent, we can just add the variances: $\mathrm{Var}(Y) = \mathrm{Var}(X_1 + X_2 + X_3) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \mathrm{Var}(X_3) = 900 + 900 + 900 = 2700$, so $\sigma_Y = \sqrt{2700} = 51.96$ minutes.

2. *Method #1:* We notice that X is Gamma with $1/\lambda = 3$ and $r = 2$, so the density of X is $f_X(x) = \frac{(1/3)^2}{\Gamma(2)} x^{2-1} e^{-x/3} = \frac{1}{9} x e^{-x/3}$ for $x > 0$, and $f_X(x) = 0$ otherwise.

Method #2: The CDF of X, for $a > 0$, is $P(X \le a) = \int_0^a \int_0^{a-x} \frac{1}{3} e^{-x/3} \frac{1}{3} e^{-y/3}\,dy\,dx = \int_0^a \frac{1}{3} e^{-x/3} \left(1 - e^{-(a-x)/3}\right)\,dx = \int_0^a \left(\frac{1}{3} e^{-x/3} - \frac{1}{3} e^{-a/3}\right)\,dx = \left(-e^{-x/3} - \frac{1}{3} e^{-a/3} x\right)\Big|_{x=0}^{a} = 1 - e^{-a/3} - \frac{1}{3} e^{-a/3} a$. Thus, $F_X(x) = 1 - e^{-x/3} - \frac{1}{3} e^{-x/3} x$ for $x > 0$, and $F_X(x) = 0$ otherwise. Differentiating with respect to x, we get $f_X(x) = \frac{1}{3} e^{-x/3} + \frac{1}{9} e^{-x/3} x - \frac{1}{3} e^{-x/3} = \frac{1}{9} e^{-x/3} x$ for $x > 0$, and $f_X(x) = 0$ otherwise.

3. *Method #1:* Since X is a Gamma random variable with $1/\lambda = 20$ and $r = 3$, then $\mathrm{Var}(X) = r/\lambda^2 = 1200$.

Method #2: Since the waiting times are independent, we can just add the variances: $\mathrm{Var}(X) = 400 + 400 + 400 = 1200$.

4. a. *Method #1:* We can just compute, treating Y as a function of X. We have $\mathbb{E}(Y) = \int_0^5 (0) \frac{1}{3} e^{-x/3} dx + \int_5^\infty (x-5) \frac{1}{3} e^{-x/3} dx = \int_5^\infty (x-5) \frac{1}{3} e^{-x/3} dx = \int_0^\infty x \frac{1}{3} e^{-(x+5)/3} dx$. We can factor out $e^{-5/3}$, so $\mathbb{E}(Y) = e^{-5/3} \int_0^\infty x \frac{1}{3} e^{-x/3}\,dx$, but the integral is 3, so $\mathbb{E}(Y) = 3e^{-5/3} = 0.5666$.

Method #2: The probability that $X \le 5$ is $1 - e^{-5/3}$, and in this case, $Y = 0$. On the other hand, the probability that $X > 5$ is $e^{-5/3}$, and we know that, given $X > 5$, it follows that the conditional distribution of $X - 5$ is exponential with expected value 3. Thus $Y = X - 5$ has expected value 3 in this case. So the expected value of Y is $\mathbb{E}(Y) = (0)(1 - e^{-5/3}) + (3)(e^{-5/3}) = 3e^{-5/3} = 0.5666$.

b. *Method #1:* We can just compute, treating Y^2 as a function of X. We have $\mathbb{E}(Y^2) = \int_0^5 (0)^2 \frac{1}{3} e^{-x/3}\,dx + \int_5^\infty (x-5)^2 \frac{1}{3} e^{-x/3}\,dx = \int_5^\infty (x-5)^2 \frac{1}{3} e^{-x/3}\,dx = \int_0^\infty x^2 \frac{1}{3} e^{-(x+5)/3}\,dx$. We can factor out $e^{-5/3}$, so $\mathbb{E}(Y^2) = e^{-5/3} \int_0^\infty x^2 \frac{1}{3} e^{-x/3}\,dx$, but the integral is $2/\lambda^2 = 2(3^2) = 18$ (i.e., the second moment of an exponential, as on page 459), so $\mathbb{E}(Y^2) = 18e^{-5/3}$. So the variance of Y is $\mathrm{Var}(Y) = 18e^{-5/3} - (3e^{-5/3})^2 = 18e^{-5/3} - 9e^{-10/3} = 3.0787$.

Method #2: The probability that $X \le 5$ is $1 - e^{-5/3}$, and in this case, $Y^2 = 0$. On the other hand, the probability that $X > 5$ is $e^{-5/3}$, and we know that, given $X > 5$, it follows that the conditional distribution of $X - 5$ is exponential with expected value 3. Thus $Y = X - 5$ has $\mathbb{E}(Y^2) = 2/\lambda^2 = 2(3^2) = 18$ in this case. So the expected value of Y^2 is $\mathbb{E}(Y^2) = (0)(1 - e^{-5/3}) + (18)(e^{-5/3}) = 18e^{-5/3}$, and the variance of Y is $\mathrm{Var}(Y) = 18e^{-5/3} - (3e^{-5/3})^2 = 18e^{-5/3} - 9e^{-10/3} = 3.0787$.

5. For $a > 0$, we have $P(X > a) = (e^{-a/5})^3 = e^{-(3/5)a}$. Thus $F_X(x) = 1 - e^{-(3/5)x}$ for $x > 0$ and $F_X(x) = 0$ otherwise. So X is exponential with $\mathbb{E}(X) = 5/3$.