STAT/MA 41600
Practice Problems: November 24, 2014
Solutions by Mark Daniel Ward

1. *Method #1:* Since X is hypergeometric with $M = 8$, $N = 11$, and $n = 2$, then $\mathrm{Var}(X) = n\frac{M}{N}\left(1 - \frac{M}{N}\right)\frac{N-n}{N-1} = 2\frac{8}{11}\left(1 - \frac{8}{11}\right)\frac{11-2}{11-1} = 216/605$.

Method #2: The mass of X is $p_X(0) = \binom{3}{2}/\binom{11}{2} = 3/55$; $p_X(1) = \binom{3}{1}\binom{8}{1}/\binom{11}{2} = 24/55$; $p_X(2) = \binom{8}{2}/\binom{11}{2} = 28/55$. Thus $\mathbb{E}(X) = (0)(3/55) + (1)(24/55) + (2)(28/55) = 16/11$, and $\mathbb{E}(X^2) = (0^2)(3/55) + (1^2)(24/55) + (2^2)(28/55) = 136/55$, so $\mathrm{Var}(X) = 136/55 - (16/11)^2 = 216/605$.

Method #3: Using the methods of Chapter 42, we write X_1 to indicate if Alice gets lemonade, and X_2 to indicate if Bob gets lemonade. So X_1 and X_2 are dependent Bernoulli's. Thus $\mathbb{E}(X_1) = \mathbb{E}(X_2) = 8/11$, and $\mathrm{Var}(X_1) = \mathrm{Var}(X_2) = (8/11)(3/11) = 24/121$. Also $\mathrm{Var}(X) = \mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + 2\,\mathrm{Cov}(X_1, X_2)$. We know $\mathrm{Cov}(X_1, X_2) = \mathbb{E}(X_1X_2) - \mathbb{E}(X_1)\mathbb{E}(X_2) = \mathbb{E}(X_1X_2) - (8/11)(8/11) = \mathbb{E}(X_1X_2) - 64/121$. Also, X_1X_2 is 0 or 1, so X_1X_2 is Bernoulli, so $\mathbb{E}(X_1X_2) = P(X_1X_2 = 1) = P(X_1 = 1 \text{ and } X_2 = 1) = P(X_1 = 1)P(X_2 = 1 \mid X_1 = 1) = (8/11)(7/10) = 28/55$. So, altogether, we have $\mathrm{Var}(X) = 24/121 + 24/121 + 2(28/55 - 64/121) = 216/605$.

2. As in the "Method #3" solution from question 1, we have $\mathrm{Cov}(X_1, X_2) = \mathbb{E}(X_1X_2) - \mathbb{E}(X_1)\mathbb{E}(X_2) = 28/55 - (8/11)(8/11) = -12/605$. Also $\mathrm{Var}(X_1) = \mathrm{Var}(X_2) = 24/121$. Thus $\rho(X_1, X_2) = \frac{\mathrm{Cov}(X_1,X_2)}{\sqrt{\mathrm{Var}(X_1)\,\mathrm{Var}(X_2)}} = \frac{-12/605}{\sqrt{(24/121)(24/121)}} = -1/10$.

3a. We have $\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$. Since X is uniform on $[10, 14]$, then $\mathbb{E}(X) = 12$. Since Y is uniform on $[22, 30]$, then $\mathbb{E}(Y) = 26$. Also $\mathbb{E}(XY) = \int_{10}^{14} x(2x+2)\frac{1}{4}\,dx = 944/3$. Thus $\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 944/3 - (12)(26) = 8/3$.

b. Since X is uniform on $[10, 14]$, then $\mathrm{Var}(X) = (14 - 10)^2/12 = 4/3$. Since Y is uniform on $[22, 30]$, then $\mathrm{Var}(Y) = (30 - 22)^2/12 = 16/3$. So $\rho(X, Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \frac{8/3}{\sqrt{(4/3)(16/3)}} = 1$.

4a. We have $\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$. Since X is uniform on $[3, 6]$, then $\mathbb{E}(X) = 4.5$. We already calculated $\mathbb{E}(Y) = 20$ in part c and part d of question 3 on problem set 35. Finally, we need $\mathbb{E}(XY) = \int_3^6 x(x^2 - 1)\frac{1}{3}\,dx = \int_3^6 x(x^2 - 1)\frac{1}{3}\,dx = 387/4$. Thus $\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 387/4 - (4.5)(20) = 27/4$.

b. Since X is uniform on $[3, 6]$, then $\mathrm{Var}(X) = (6 - 3)^2/12 = 3/4$.
Also $\mathbb{E}(Y^2) = \int_3^6 (x^2 - 1)^2(1/3)\,dx = 2306/5$. So $\mathrm{Var}(Y) = 2306/5 - 20^2 = 306/5$.
Thus $\rho(X, Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \frac{27/4}{\sqrt{(3/4)(306/5)}} = 0.996$.

5. The mass of X is $p_X(1) = 7/16$; $p_X(2) = 5/16$; $p_X(3) = 3/16$; $p_X(4) = 1/16$; so $\mathbb{E}(X) = (1)(7/16) + (2)(5/16) + (3)(3/16) + (4)(4/16) = 15/8$.

The mass of Y is $p_Y(1) = 1/16$; $p_Y(2) = 3/16$; $p_Y(3) = 5/16$; $p_Y(4) = 7/16$; so $\mathbb{E}(Y) = (1)(1/16) + (2)(3/16) + (3)(5/16) + (4)(7/16) = 25/8$.

The expected value of XY is

$$\begin{aligned}
\mathbb{E}(XY) &= \frac{1}{16}((1)(1) + (1)(2) + (1)(3) + (1)(4) \\
&\quad + (1)(2) + (2)(2) + (2)(3) + (2)(4) \\
&\quad + (1)(3) + (2)(3) + (3)(3) + (3)(4) \\
&\quad + (1)(4) + (2)(4) + (3)(4) + (4)(4)) \\
&= (1/16)(1 + 2 + 3 + 4 + 2 + 4 + 6 + 8 + 3 + 6 + 9 + 12 + 4 + 8 + 12 + 16) \\
&= 25/4
\end{aligned}$$

Thus $\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 25/4 - (15/8)(25/8) = 25/64$.