STAT/MA 41600
Practice Problems: December 1, 2014
Solutions by Mark Daniel Ward

1. *Method #1:* Since X, Y have a joint uniform distribution on the triangle, then given $X = 1/2$, we know that Y is uniformly distributed on $[0, 3/2]$. Thus the conditional expectation of Y given $X = 1/2$ is exactly $\mathbb{E}(Y \mid X = 1/2) = \frac{3/2+0}{2} = 3/4$.

Method #2: The area of the triangle is 2, so $f_{X,Y}(x, y) = 1/2$ on the triangle. Also $f_X(1/2) = \int_0^{3/2} f_{X,Y}(x, y)\,dy = \int_0^{3/2} 1/2\,dy = 3/4$. So $f_{Y|X}(y \mid 2) = \frac{f_{X,Y}(2,y)}{f_X(2)} = \frac{1/2}{3/4} = 2/3$. Thus $\mathbb{E}(Y \mid X = 2) = \int_0^{3/2}(y)(2/3)\,dy = 3/4$.

2. a. *Method #1:* Given $X = 3$, there are 7 equally-likely outcomes: $(3, 3), (3, 4), (3, 5), (3, 6), (4, 3), (5, 3), (6, 3)$. So $\mathbb{E}(Y \mid X = 3) = \frac{1}{7}(3+4+5+6+4+5+6) = \frac{33}{7}$.

Method #2: We have $p_X(3) = 7/36$. Also $p_{X,Y}(3, 3) = 1/36$, and $p_{X,Y}(3, y) = 2/36$ for $y = 4, 5, 6$. Also $p_{Y|X}(y \mid 3) = \frac{p_{X,Y}(3,y)}{p_X(3)}$. So $p_{Y|X}(y \mid 3) = \frac{1/36}{7/36} = 1/7$, and $p_{Y|X}(y \mid 3) = \frac{2/36}{7/36} = 2/7$, for $y = 4, 5, 6$. So $\mathbb{E}(Y \mid X = 3) = (3)(1/7) + (4)(2/7) + (5)(2/7) + (6)(2/7) = 33/7$.

b. We have $\mathbb{E}(X + Y \mid X = 3) = \mathbb{E}(X \mid X = 3) + \mathbb{E}(Y \mid X = 3) = 3 + 33/7 = 54/7$.

3. We know that Y is a Gamma random variable with $\lambda = 1$ and $r = 2$. Thus $f_Y(3) = e^{-3}3/(2!) = 3e^{-3}$. Also $f_{X_1,Y}(x, 3) = f_{Y|X_1}(3 \mid x)f_{X_1}(x)$. Of course $f_{X_1}(x) = e^{-x}$. For any $y > X_1$, we have $F_{Y|X_1}(y \mid x) = P(Y < y \mid X_1 = x) = P(Y - x < y - x \mid X_1 = x) = P(X_2 < y-x) = F_{X_2}(y-x)$. Differentiating with respect to y gives $f_{Y|X_1}(y \mid x) = f_{X_2}(y-x) = e^{-(y-x)}$. So $f_{Y|X_1}(3 \mid x) = e^{-(3-x)}$ for $3 > x$. So $f_{X_1,Y}(x, 3) = f_{Y|X_1}(3 \mid x)f_{X_1}(x) = e^{-(3-x)}e^{-x} = e^{-3}$. So $f_{X_1 \mid Y}(x_1 \mid 3) = \frac{f_{X_1,Y}(x,3)}{f_Y(3)} = \frac{e^{-3}}{3e^{-3}} = 1/3$. Thus the conditional density of X_1, given $Y = 3$, is uniform on $[0, 3]$. So $\mathbb{E}(X_1 \mid Y = 3) = 3/2$. I.e., $\mathbb{E}(X_1 \mid Y = 3) = \int_0^3(x)(1/3)\,dx = 3/2$.

4. a. If Bob gets lemonade, then Alice has 10 remaining drinks, of which 7 are lemonade, so $\mathbb{E}(X_1 \mid X_2 = 1) = P(X_1 = 1 \mid X_2 = 1) = 7/10$.

b. If Bob does not get lemonade, then Alice has 10 remaining drinks, of which 8 are lemonade, so $\mathbb{E}(X_1 \mid X_2 = 0) = P(X_1 = 1 \mid X_2 = 0) = 8/10$.

5. *Method #1:* Given that there are 12 roses, then each is equally likely to have been picked by Sally or David, so for each flower, we expect it was picked by Sally half the time or by David half the time. So the expected number of roses picked by Sally is 6.

More formally, to see the argument in Method #1, let $X_1, \ldots, X_{10}$ be indicators for whether the 1st, 2nd, ..., 10th flower of Sally is a rose. Then $X = X_1 + \cdots + X_{10}$, so

$$\begin{aligned}\mathbb{E}(X \mid Y = 12) &= \mathbb{E}(X_1 + \cdots + X_{10} \mid Y = 12)\\ &= \mathbb{E}(X_1 \mid Y = 12) + \cdots + \mathbb{E}(X_{10} \mid Y = 12)\\ &= 12/20 + \cdots + 12/20\\ &= (10)(12/20)\\ &= 6.\end{aligned}$$

Method #3: We can go through the same kind of argument as in the cookie example in the Conditional Expectation chapter of the book, using 10 flowers per person instead of 5 cookies per person, and using $Y = 12$ instead of $Y = 7$. We will get $p_{X|Y}(x \mid 12) = \frac{\binom{10}{x}\binom{10}{12-x}}{\binom{20}{12}}$. So, conditioned on $Y = 12$, we see that X is hypergeometric with $M = 10$ flowers for Sally, and $N = 20$ flowers altogether, and $n = 12$ of the flowers are selected to be designated as roses. Thus $\mathbb{E}(X \mid Y = 12) = nM/N = (12)(10)/20 = 6$.