1. a. Since $X$ is uniform and $Y$ has the form $Y = aX + b$ for constants $a, b$, then $Y$ is uniform too. Notice $2(10) + 2 \leq Y \leq 2(14) + 2$, i.e., $22 \leq Y \leq 30$. Also $\frac{1}{30-22} = \frac{1}{8}$. So $f_Y(y) = \frac{1}{8}$ for $22 \leq Y \leq 30$, and $f_Y(y) = 0$ otherwise.

b. Since the density of $Y$ is constant on $[22, 30]$, then $P(Y > 28) = \frac{\text{length of } [28, 30]}{\text{length of } [22, 30]} = \frac{2}{8} = 1/4$.

c. We have $P(Y > 28) = P(2X + 2 > 28) = P(2X > 26) = P(X > 13)$. Since the density of $X$ is constant on $[10, 14]$, then $P(X > 13) = \frac{\text{length of } [13, 14]}{\text{length of } [10, 14]} = 1/4$.

2. a. Since $X$ is uniform and $Y$ has the form $Y = aX + b$ for constants $a, b$, then $Y$ is uniform too. Notice $(1.07)(4) + 3.99 \leq Y \leq (1.07)(9) + 3.99$, i.e., $8.27 \leq Y \leq 13.62$. Also $\frac{1}{13.62 - 8.27} = \frac{1}{5.35}$. So $f_Y(y) = \frac{1}{5.35}$ for $8.27 \leq Y \leq 13.62$, and $f_Y(y) = 0$ otherwise.

b. Method #1: Since $Y$ is uniform on $[8.27, 13.62]$, the expected value of $Y$ is the midpoint of the interval, i.e., $E(Y) = \frac{8.27 + 13.62}{2} = 10.945$.

Method #2: We calculate $E(Y) = \int_{8.27}^{13.62} y \frac{1}{5.35} dy = \frac{13.62^2 - 8.27^2}{2} \frac{1}{5.35} = 10.945$.

c. We calculate $E(X) = \int_{4}^{9} (1.07x + 3.99) \frac{1}{5} dx = \frac{1}{5} (1.07x^2/2 + 3.99x)\bigg|_{x=4}^{9} = 10.945$.

3. a. For $8 \leq a \leq 35$, we have $P(Y \leq a) = P((X - 1)(X + 1) \leq a) = P(X^2 - 1 \leq a) = P(X^2 \leq a + 1) = P(X \leq \sqrt{a + 1}) = \frac{\sqrt{a+1}-3}{6-3}$. Thus, the CDF of $Y$ is

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 8, \\ \frac{\sqrt{y+1}-3}{3} & \text{if } 8 \leq y \leq 35, \\ 1 & \text{if } 35 < y. \end{cases}$$

b. For $8 \leq y \leq 35$, we differentiate $F_Y(y)$ with respect to $y$, and we get $f_Y(y) = \frac{1}{6} (y+1)^{-1/2}$; otherwise, $f_Y(y) = 0$.

c. We use $u = y + 1$ and $du = dy$ to compute $E(Y) = \int_{8}^{35} y \frac{1}{6} (y + 1)^{-1/2} dy = \int_{9}^{36} \frac{1}{6} (u - 1)u^{-1/2} du = \int_{9}^{36} \frac{1}{6} (u^{3/2} - u^{-1/2}) du = \frac{1}{6} (\frac{2}{3} u^{3/2} - 2 u^{1/2})\bigg|_{u=9}^{36} = \frac{1}{6} ((2/3)(216) - (2)(6)) = ((2/3)(27) - (2)(3)) = 20$.

d. We compute $E((X-1)(X+1)) = \int_{3}^{6} (x-1)(x+1)(1/3) dx = \int_{3}^{6} (1/3)(x^2 - 1) dx = (1/3)(\frac{1}{3}x^3 - x)\bigg|_{x=3}^{6} = (1/3)((72 - 6) - (9 - 3)) = 20$.

4. We have $\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$. Since $X, Y$ have a joint uniform distribution on a triangle with area $(2)(2)/2 = 2$, then $f_{X,Y}(x,y) = 1/2$ on the triangle, and $f_{X,Y}(x,y) = 0$.
otherwise. So:

\[ E(XY) = \int_0^2 \int_0^{2-x} xy \frac{1}{2} \, dy \, dx = \frac{1}{3}, \]

and

\[ E(X) = \int_0^2 \int_0^{2-x} x \frac{1}{2} \, dy \, dx = \frac{2}{3}, \]

and (since everything is symmetric, we don’t even need to calculate):

\[ E(Y) = \int_0^2 \int_0^{2-x} y \frac{1}{2} \, dy \, dx = \frac{2}{3}. \]


5. Since \(X_j\) is Bernoulli with \(p = 2/19\), then \(\text{Var}(X_j) = (2/19)(17/19) = 34/361\).

Also Cov\((X_i, X_j) = E(X_iX_j) - E(X_i)E(X_j)\). Also \(E(X_i) = 2/19\) and \(E(X_j) = 2/19\), so we only need \(E(X_iX_j)\). Notice \(X_iX_j\) is 0 or 1, i.e., the product \(X_iX_j\) is Bernoulli, so \(E(X_iX_j) = P(X_iX_j = 1)\). [We can also see this by \(E(X_iX_j) = 1P(X_iX_j = 1) + 0P(X_iX_j = 0) = P(X_iX_j = 1)\).]

Now we use \(P(X_iX_j = 1) = P(X_i = 1 \text{ and } X_j = 1) = P(X_i = 1)P(X_j = 1 | X_i = 1)\). We know \(P(X_i = 1) = 2/19\). Once \(X_i = 1\) is given, there is a row of 18 open seats where the \(j\)th couple might sit. The man sits on the end with probability 2/18 and his wife beside him with probability 1/17, or the man does not sit on the end, with probability 16/18 and his wife beside him with probability 2/17, so \(P(X_j = 1 | X_i = 1) = (2/18)(1/17) + (16/18)(2/17) = 1/9\). [Alternatively, this can be calculated by observing that there are (18)(17) places that they can sit, but there are 17 adjacent pairs of seats, and they can sit in them 2 ways, so \(P(X_j = 1 | X_i = 1) = \frac{(17)(2)}{(18)(17)} = 2/18 = 1/9\).]

So \(E(X_iX_j) = P(X_iX_j = 1) = (2/19)(1/9)\).

So Cov\((X_i, X_j) = E(X_iX_j) - E(X_i)E(X_j) = (2/19)(1/9) - (2/19)(2/19) = 2/3249\).

Finally \(\text{Var}(X) = \sum_{j=1}^{10} \text{Var}(X_j) + 2 \sum_{1 \leq i < j \leq 10} \text{Cov}(X_i, X_j) = (10)(34/361) + (90)(2/3249) = 360/361\).